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TOPOLOGY

Apfelbacher, Karl. Über Beziehungen zwischen Umgebungsräumen und Häufungsräumen. Monatsh. Math. Phys. 49, 153–193 (1940). [MF 2993]

A neighborhood space (Umgebungsräum) in the sense of this article is an arbitrary class (Menge) $\mathfrak{M}$ such that for every element P and subset $\mathfrak{A}$ of $\mathfrak{M}$ it is determined whether $\mathfrak{A}$ is or is not a neighborhood (Umgebung) of P. Also, a limiting point space (Haufungsräum) $\mathfrak{H}$ is a set such that for every point P and subset $\mathfrak{A}$ of $\mathfrak{H}$ it is determined whether P is or is not a limiting point (Haufungspunkt) of $\mathfrak{A}$. A point P of a neighborhood space $\mathfrak{M}$ is a limiting point of a set $\mathfrak{A}$ if each neighborhood of P contains at least one point of $\mathfrak{A}-\{P\}$. Every subset of a limiting point space $\mathfrak{H}$ is a neighborhood of a point P if P is not a limiting point of its complement with respect to $\mathfrak{H}$. If $\mathfrak{T}_0$ is an arbitrary neighborhood space, the foregoing definition determines a limiting point space denoted by $\mathfrak{H}(\mathfrak{T}_0)$. From a limiting point space $\mathfrak{H}_0$ we derive similarly a neighborhood space $\mathfrak{T}(\mathfrak{H}_0)$. For any $\mathfrak{T}_0$ the space $\mathfrak{T}_1=\mathfrak{T}(\mathfrak{H}(\mathfrak{T}_0))$ is identical to $\mathfrak{T}_0$ if and only if: (t_1) each $\mathfrak{T}_0$-neighborhood of a point P contains P; (t_2) every set which contains a neighborhood of a point P in the space $\mathfrak{T}_0$ is a $\mathfrak{T}_0$-neighborhood of P. The corresponding condition for the identity of $\mathfrak{H}_0$ and $\mathfrak{H}_1=\mathfrak{H}_1(\mathfrak{T}(\mathfrak{H}_0))$ is: (h_1) if P is a limiting point of a set $\mathfrak{A}$ in the space $\mathfrak{H}_0$, then P is also a limiting point in $\mathfrak{H}_0$ of every set $\mathfrak{B}\supseteq\mathfrak{A}-\{P\}$. The space $\mathfrak{H}(\mathfrak{T}_0)$ always has the property (h_1). If $\mathfrak{H}_0$ is a limiting point space with the property (h_1) there is always at least one neighborhood space $\mathfrak{T}_0$ such that $\mathfrak{H}_0=\mathfrak{H}(\mathfrak{T}_0)$. The space $\mathfrak{T}(\mathfrak{H}_0)$ always has the property (t_1). Conversely, if $\mathfrak{T}_0$ has the property (t_1), it is a $\mathfrak{T}(\mathfrak{H}_0)$. The space $\mathfrak{T}(\mathfrak{H}_0)$ has the property (t_2) if and only if the space $\mathfrak{H}_0$ has the property: if a point P is an $\mathfrak{H}_0$-limiting point of a set $\mathfrak{A}$ which does not contain P, then P is an $\mathfrak{H}_0$-limiting point of every set $\mathfrak{B}\supseteq\mathfrak{A}$ which does not contain P.

A neighborhood space $\mathfrak{T}$ and a limiting point space $\mathfrak{H}_0$ are equivalent if $\mathfrak{T}_0=\mathfrak{T}(\mathfrak{H}_0)$ and $\mathfrak{H}_0=\mathfrak{H}(\mathfrak{T}_0)$. A space $\mathfrak{T}_0$ is equivalent to a space $\mathfrak{H}_0$ if and only if it has properties (t_1), (t_2). The space $\mathfrak{H}_0$ is uniquely determined and identical to the space $\mathfrak{H}(\mathfrak{T}_0)$. It has the property ($h_1$). A corresponding theorem holds for limiting point spaces. A number of independence examples are given. The definition of limiting point given above is compared with those of Vietoris and Gordon. Equivalent sets of axioms for neighborhood and limiting point spaces are formulated in terms of the properties (t_1), (t_2), (h_1) and other well-known properties of topological spaces. *E. W. Chittenden* (Iowa City, Iowa).

Lubben, R. G. Separabilities of arbitrary orders and related properties. Bull. Amer. Math. Soc. 46, 913–919 (1940). [MF 3448]

For a cardinal α the author defines strong α-separability, limit point of order α, α-isolated point, almost perfect α-compactness and other terms for sets in an abstract space. He then establishes the equivalence of statements involving these terms for a space V Fréchet in which the operation of derivation is distributive and α is a regular cardinal. A suggestion of the nature of the questions considered may be obtained if one knows that they grew from the embryo of the equivalence in a metric space of the two statements (1) each point set is separable; (2) each uncountable point set contains a condensation point of itself. The author's work here considerably extends his earlier work for a space H Fréchet and he shows how it is related to and supplements some work by Appert. *J. F. Randolph.*

Price, G. Baley. A generalization of a metric space with applications to spaces whose elements are sets. Amer. J. Math. 63, 46–56 (1941). [MF 3628]

Distance functions $d(x, y)$ are studied whose values are elements of a partially ordered set D in which a commutative addition operation is defined. It is assumed that $d(x, y)$ is symmetric and satisfies the triangle inequality. A spherical neighborhood of a point a consists of all points x such that $d(x, a)<\epsilon$, where ϵ is any element of a special subset E of "positive" elements of D. Various conditions on the sets D and E are shown to be sufficient to insure that the resulting topological space K is a regular Hausdorff space, or a metrizable space, or that compact sets in K are totally bounded. The necessity of these conditions is not discussed.

As an example, the author considers a partially ordered linear space K. Here the sets D and E consist of all elements of K and all positive elements of K, respectively, and $d(x, y)$ is defined to be the absolute value element $|x-y|$. The resulting topology is not very useful, since unions of closed sets may not be closed. Another application is to Borel fields of sets. Here the set D is the field itself, the special subset E may be any collection of nonempty sets whose intersection is empty, and $d(x, y)$ is the symmetric difference of x and y. Set union is taken as the addition operation in D. It is shown that the resulting space is complete, and that it is metrizable if E is a decreasing sequence of sets. Finally it is shown that if M is a Borel field in which an exterior measure is defined, defining $d(x, y)$ to be the exterior measure of the symmetric difference of x and y makes M a complete metric space, which, in case M is the field of measurable subsets of the unit interval, is also convex in the sense of Menger and quasiconvex in the sense of Blanc. *O. Frink.*

Alexandroff, P. S. Correction to a paper by P. S. Alexandroff and V. V. Niemytzki: Condition of metrization of topological spaces and the axiom of symmetry. Rec. Math. [Mat. Sbornik] N.S. 8 (50), 519 (1940). (Russian) [MF 3747]

The paper cited appeared in Rec. Math. [Mat. Sbornik] N.S. 3 (45), 663–671 (1938); note especially pp. 666–667.

Alexandroff, P. Über die Dimension der bikompakten Räume. C. R. (Doklady) Acad. Sci. URSS (N.S.) 26, 619–622 (1940). [MF 3569]

The following simple lemma is fundamental in this paper: if A is a closed, U an open subset of a normal space R and

$A \subset U$, there exists an A_1 which is both closed and a G_δ such that $A \subset A_1 \subset U$. This has the interesting consequence that a basis of open sets for the space may be chosen from among those open sets which are also F_σ. It leads, also, to the result that R admits an ω_R-map into the nerve of ω_R (finite covering by open sets) regarded as a polyhedron. The principal results characterize the dimension of a bicompact R as the smallest n such that for every ω_R there is an ω_R-map into an n-dimensional polyhedron, and also as the largest n for which there exists an essential map of R onto an n-dimensional simplex. Part of this generalizes a result of Vedenisoff [Compositio Math. 7, 194–200 (1939); these Rev. 1, 107] where the extra condition is imposed that every closed set be a G_δ. Finally, a generalization to bicompact R is given of the Menger-Nöbeling imbedding theorem.

L. Zippin (Flushing, N. Y.).

Erdös, Paul. The dimension of the rational points in Hilbert space. Ann. of Math. (2) **41**, 734–736 (1940). [MF 3014]

The author shows that the set R of points of the real Hilbert-space having all coordinates rational is of dimension 1. The same holds for the set R_0 of points of H of the form $(1/n_1, 1/n_2, \cdots)$, where n_r are positive integers. Also the closure $R_1 = \bar{R}_0$ has dimension 1. This result is remarkable in view of the fact that the sets corresponding to R, R_0 and R_1 in a finite dimensional Euclidean space are of dimension zero. The set R_1 is a complete separable metric space with the property $\dim R_1 \cdot R_1 = \dim R_1 = 1$. It should also be noticed that R_1 is disconnected between any two of its points.

W. Hurewicz (Chapel Hill, N. C.).

Šura-Bura, M. R. Espaces bicompacts comme images des discontinus. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 431–435 (1940). [MF 3220]

The author claims to prove the following theorem: Every closed subset F of a discontinuum D (that is, the product of discrete spaces each containing two points) is a continuous image of D. His proof fails at line 25 of page 435, for the transformation involved need not be open on F and its closed subsets, although it is in fact open on D. The theorem is known when D is a countable product; the extension to the general case is untrue. For those points of D with at most one coordinate equal to 1 form a closed set F which contains, if there are uncountably many factors, a family of uncountably many disjoint open sets. Since D does not contain uncountably many disjoint open sets, F is not a continuous image of D.

J. W. Tukey (Princeton, N. J.).

Betz, Ebon E. Accessibility and separation by simple closed curves. Amer. J. Math. **63**, 127–135 (1941). [MF 3636]

This paper is another step towards the solution of an outstanding problem proposed by J. R. Kline, namely the determination of whether or not the following two conditions placed on a compact locally connected continuum M are sufficient to insure that M is a simple closed surface: (a) no two points of M separate M; (b) every simple closed curve J in M separates M. The author strengthens (b) by demanding (b′) for every J in M the number of components of $M-J$ is at least two and at most finite. He proves that (a) and (b′) together imply that for any J in M and any component D of $M-J$ every boundary point of D in M is regularly accessible from D. The author shows that by using this theorem one can obtain an exceedingly simple proof of the theorem of Zippin [Amer. J. Math. **52**, 348 (1930)] that M is a topological sphere provided it satisfies (a) and $M-J$ has exactly two components for every simple closed curve J in M.

D. W. Hall (Providence, R. I.).

Harrold, O. G., Jr. Continua of finite degree and certain product sets. Bull. Amer. Math. Soc. **46**, 951–953 (1940). [MF 3454]

It is shown that either of the following two properties for a continuum M is equivalent to the condition that M be a continuum of finite degree: (1) M is locally connected and to each pair of closed, disjoint subsets A and B in M there corresponds a finite collection of disjoint perfect sets such that any continuum in M intersecting both A and B contains at least one of these perfect sets; (2) if K and K_i $(i=1, 2, \cdots)$ are nondegenerate continua in M with $\lim K_i = K$, there exists an integer n such that the common part of all the sets K_i is an uncountable set.

G. T. Whyburn (Charlottesville, Va.).

Youngs, J. W. T. Arc-spaces. Duke Math. J. **7**, 68–84 (1940). [MF 3390]

In discussing the structure of continuous curves, it is concluded that any continuous curve has a "dendritic" structure with respect to its cyclic elements. An objection to this is that it is usually impossible to consider the space whose elements are the cyclic elements of the given curve as a topological space in any usual sense, although R. L. Moore [Rice Inst. Pamphlets **23**, no. 1 (1936)] has considered spaces which include spaces of the above type. The author develops a theory of arc spaces to replace Peano spaces in the sense that cyclic elements can be defined in them. They have the further property that the fundamental arc space relations may be defined among the cyclic elements of an arc space E so that the hyperspace of the cyclic elements of E constitutes another arc space. Let E be a set of elements a, b, c, etc., and K be a class of ordered subsets $\alpha(a, b)$ of E, called "arcs," where $\alpha(a, b)$ has a first element a and last element b. The set E together with the class K form an arc space if the following axioms hold: (A1) if $\alpha(a, b) \in K$, then $\alpha(b, a) \in K$; (A2) if $x < y$ on $\alpha(a, b)$, then $\alpha(x, y)$ (=all z of $\alpha(a, b)$ with $x \leqq z \leqq y$) $\in K$ and is the only subarc of $\alpha(a, b)$ from x to y; (A3) the Dedekind cut axiom holds for each arc; (A4) if $\alpha_1(a, x)$ and $\alpha_2(x, b)$ have only x in common, their sum is an arc; (A5) if $\alpha_1(a, b)$ and $\alpha_2(c, b)$ are two arcs, the set common to them has a first element along $\alpha(a, b)$; (A6) any two points of E are end points of some arc.

C. B. Morrey, Jr. (Berkeley, Calif.).

Markouchevitch, A. Sur certaines classes de transformations continues. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 301–304 (1940). [MF 3330]

The following results are proved: Let $P' = f(P)$ be a continuous transformation defined on a domain G (in n-space) in which there exists a set Z of measure zero such that, if P_0 is not in Z, then (a) the transformation is 1-1 in a neighborhood $U(P_0)$ of P_0 and (b) there exists in $U(P_0)$ a sequence $\{\Gamma_i(P_0)\}$ of hypersurfaces, each homeomorphic to the surface of an n-sphere, satisfying the conditions

$$\text{(A)} \quad \lim_{i\to\infty} \frac{r_{i+1}(P_0)}{r_i(P_0)} > 0, \quad \lim_{i\to\infty} \frac{r_i(P_0)}{R_i(P_0)} > 0, \quad \lim_{i\to\infty} \frac{r_i'(P_0')}{R_i'(P_0')} > 0,$$

$$\lim_{i\to\infty} R_i(P_0) = 0,$$

where $r_i(P_0)$ and $R_i(P_0)$ denote the minimum and maximum distances of $\Gamma_i(P_0)$ from P_0 and $r_i'(P_0')$ and $R_i'(P_0')$ denote

the minimum and maximum distances of the transformed surfaces $\Gamma_i'(P_0')$ from the point $P_0'=f(P_0)$. Then the coordinate functions each possess total differentials (in the sense of Stolz) almost everywhere. If, further, there exists a family $\{\Gamma_\lambda(P_0)\}$ $(0<\lambda<\infty)$ which fills out a subneighborhood of $U(P_0)$ (excepting P_0) and satisfies the last three conditions (A) with i replaced by λ, and if the transformation carries sets of positive measure into sets of positive exterior measure, then the Jacobian is different from zero almost everywhere. *C. B. Morrey* (Berkeley, Calif.).

Wallace, A. D. Quasi-monotone transformations. Duke Math. J. **7**, 136–145 (1940). [MF 3394]

A continuous mapping $T(A)=B$ of two locally connected continua is called quasi-monotone if, for any connected set H in B with a non-void interior, each component of $T^{-1}(H)$ maps onto all of H. Many characterizations of quasi-monotone mappings are obtained; for example, a continuous $T(A)=B$ is quasi-monotone if and only if, for every connected set $Z\subset B$ and every quasi-component S of $T^{-1}(Z)$, $T(S)=Z$. Every monotone and every interior transformation is quasi-monotone. The product T_2T_1 of two quasi-monotone transformations in quasi-monotone. Every quasi-monotone T admits a factorization $T=T_2T_1$, where $T_1(A)=A'$ is monotone and $T_2(A')=B$ is light and interior. It is also proved that the property of being a rational curve or a regular curve is invariant under quasi-monotone transformations. At the end, the influence of a quasi-monotone transformation on the 1-dimensional structure of a continuum is discussed. *S. Eilenberg* (Ann Arbor, Mich.).

Schweigert, G. E. A note on the limit of orbits. Bull. Amer. Math. Soc. **46**, 963–969 (1940). [MF 3457]

In a recent paper [Duke Math. J. **4**, 719–724 (1938)] D. W. Hall and the author established the following theorem: If M is a compact metric space, $T(M)=M$ a pointwise periodic transformation, $\{G_{n_k}\}$ a convergent sequence of point-orbits under T with limit set L, then L is connected if it contains a connected subset B such that $T(B)=B$. If X is a component of an invariant set K, that is, $T(K)=K$, then X together with all its iterated images under T is called a component-orbit. In the present note it is shown that, if "point-orbits" in the above theorem are replaced by "closed component-orbits," then L is always a closed component-orbit. The above theorem follows then as an obvious corollary. The present theorem is applied to an interesting example to show that a homeomorphism may be pointwise periodic on an everywhere dense set without being pointwise periodic on the entire space. *W. L. Ayres.*

Smith, P. A. Fixed-point theorems for periodic transformations. Amer. J. Math. **63**, 1–8 (1941). [MF 3624]

Using methods recently developed by him to study homology properties of the set of fixed-points of a periodic homeomorphism T of a space M into itself [Ann. of Math. (2) **39**, 127–163 (1938); **40**, 690–711 (1939); these Rev. **1**, 30; referred to below as (1) and (2)], the author now contributes to results found, when M is a subspace of a euclidean space and T is of prime period, by him [Ann. of Math. (2) **35**, 572–578 (1934)] and S. Eilenberg [Duke Math. J. **6**, 428–437 (1940); these Rev. **1**, 319]. A Hausdorff space M is acyclic mod p if for every bicompact set A there exists a bicompact $A'\supset A$ such that mod p Čech cycles [Fund. Math. **18**, 149–183 (1932)] in A are ~ 0 in A'. Let $\alpha>0$ be an integer. The principal theorems are: (I(α)) if M is finite dimensional, locally bicompact and acyclic mod a prime p, every T of period p^α admits at least one fixed-point; (III) if M is compact the restriction of finite dimensionality may be dropped. The step from (I(1)) to (I(α)) is in both cases made by means of (II) when $\alpha=1$, the totality of fixed-points is acyclic mod p. In case M is a finite simplicial complex K, acyclic mod p, and $T=T_K$ is simplicial and such that its invariant simplexes form a subcomplex, the proofs are combinatorial using chains $\sigma C=(1+T+\cdots+T^{r-1})C$ and $\delta C=(1-T)C$ found in (1), (2), C a chain mod p. In the general case the cycles are Čech and the proofs depend on constructing, along the lines of (1), a complete system of finite coverings U_i of an invariant bicompact subset N of M such that: (a) T induces a T_K on nerve U_i, (b) dim (nerve U_i) is bounded, (c) the projection π_iC_i onto U_{i+1} of a U_i-cycle which is on a bicompact set A_i is the U_{i+1}-coordinate of a complete cycle in A_i and is ~ 0, where π_i permutes with T. Two further extensions of (1), (2) are: Every T acting on euclidean 3-space, and every regular T acting on euclidean 4-space admits a fixed-point. T, of period q, is regular if the non-empty fixed-point sets of $T^2, \cdots, T^{q-1}$ are locally euclidean. (T locally analytic implies T regular.) *W. W. Flexner* (Ithaca, N. Y.).

Eilenberg, Samuel. An invariance theorem for subsets of S^n. Bull. Amer. Math. Soc. **47**, 73–75 (1941). [MF 3813]

If A and B are homeomorphic subsets (not neccssarily closed) of an n-sphere S_n, the number of components of S_n-A, if finite, equals the number of components of S_n-B. This follows from a duality relation between $(n-1)$-cycles in A and 0-cycles in S_n-A (coefficients modulo 2). *P. A. Smith* (New York, N. Y.).

Eilenberg, Samuel. Ordered topological spaces. Amer. J. Math. **63**, 39–45 (1941). [MF 3627]

An "ordered topological space" is, in effect, a simply ordered set whose topology is obtainable by a weakening of its intrinsic topology. The author proves that a topological connected space X can be ordered if and only if the subset of its square X^2 obtained by deleting the diagonal of points (x, x) is not connected; the same condition also characterizes those connected locally connected separable topological spaces which are homeomorphic with subsets of the linear continuum. *G. Birkhoff* (Cambridge, Mass.).

Maeda, Fumitomo. Partially ordered linear spaces. J. Sci. Hirosima Univ. Ser. A. **10**, 137–150 (1940). [MF 3381]

Let L be any distributive lattice with zero element 0. The author proves that, if in L one identifies a with b whenever $a\cap x=0$ is equivalent to $b\cap x=0$, one gets a generalized Boolean algebra. This is, in a certain sense, the largest generalized Boolean algebra to which L is homomorphic. Any vector lattice is [illegible] (Freudenthal); hence the result applies to such a case. In this case, the [illegible] Freudenthal's integral repr[illegible] Nederl. Akad. Wetensch., Proc. **39**, 641–651 (1936). With the hypothesis of conditional completeness (every set has a least upper bound), the author gives a new theoretical setting.
G. Birkhoff (Cambridge, Mass.).

Lusternick, L. Intersections dans les espaces localement linéaires. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 771–774 (1940). [MF 3227]

Lusternick, L. Structure topologique d'un espace fonctionnel. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 775–777 (1940). [MF 3228]

If $f_1, f_2, \cdots, f_n$ are linearly independent linear functionals

on a linear space E, then $f_i(x)=c_i$, $i\leqq k$, determine a subspace of E of dimension $-k$. The additional restriction $f_{k+1}(x)\leqq c_{k+1}$ determines a half subspace of dimension $-k$. Subspaces of positive dimension are defined in the usual way. A metric space M is locally linear if there is a fixed Banach space E such that each $a\varepsilon M$ has a neighborhood almost isomorphic with a suitable neighborhood $N(a)$ of the identity in E. That is, a homeomorphism T of U_a on $N(a)$ exists with

$$|\rho(b, c)-\|T(b)-T(c)\|| = o(\rho(a, b)+\rho(a, c)).$$

The usual spaces of the calculus of variations are locally linear. If H is a (half) subspace of E, and T is any such homeomorphism, then $T^{-1}(H)$ is a (half) element of M of the same dimension as H. A composite element of dimension $-n$ is the union of k elements of dimension $-n$, whose intersection is an element of dimension $-kn$. A nonempty subset P of M is a manifold of dimension r if, for every $a\varepsilon M$, its intersection with every sufficiently small neighborhood of a is either (1) an element, (2) a half element, (3) a composite element of dimension r, or (4) the empty set. Its frontier consists of the points for which (2) always holds. If $r\geqq 0$, we add the condition that P be compact. A manifold without frontier is a cycle. A cycle which is a frontier bounds.

The following results are announced. (I) The intersection of cycles is a cycle. (II) The intersection of a bounding cycle of positive dimension with any cycle is a bounding cycle. Since an uneven number of points is a nonbounding cycle, (II) enables one to show that certain cycles are nonbounding. The length of a cycle is defined as the supremum of the numbers m such that there exist m cycles whose common intersection with the given cycle is nonbounding. [Typographical errors occur here in the paper.] The length of a space is the supremum of the length of its cycles. The space M_{ab} consists of all curves joining two fixed points on the two-sphere whose tangent directions and curvatures are piecewise continuous. The distance in M_{ab} is that of "proximity of the second order." The author shows that the length of M_{ab} is infinite. *J. W. Tukey* (Princeton, N. J.).

Sirvint, G. Zur Geometrie linearer Räume. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 119–122 (1940). [MF 3543]

The topology of a linear space E is studied with the help of the functional $|x/U| = \inf a$, $a>0$, $x\varepsilon aU$, where U is a convex "encirclement" (Umkreisung) of the origin and aU represents a homothetic transformation of U. A family $\mathfrak{F}$ of open sets is called a reduced family of open sets in case an arbitrary open set in the underlying topology can be constructed from the sets in $\mathfrak{F}$ by the operations of translation, magnification, set addition and finite set multiplication. The notions of convergence and boundedness can be studied by methods involving the sets in $\mathfrak{F}$. Metric spaces and normed vector spaces correspond to certain types of reduced families. Finally the role played by the category classification of the space is examined. For instance, a norm can be introduced into a space E of the second category if and only if there exists in E a convex bounded "Umkreisung." For examples, the spaces l^p and L^p, $0<p<1$, are investigated. The principal application of the theory is to the study of the topology of the first and second conjugate spaces of a given topological space [cf. the next review].

E. R. Lorch (New York, N. Y.).

Sirvint, G. Espace de fonctionnelles linéaires. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 123–126 (1940). [MF 3544]

This paper is based upon the paper reviewed above. The purpose is to lay the foundations of the theory of the first and second conjugate spaces of a given locally convex space E. A functional $\phi(x)$, $x\varepsilon E$, is called continuous if the preimage of an open set is an open set in E. A convex topology is introduced into the space E^* of linear (that is, continuous and additive) functionals by means of a reduced family of open sets consisting of the sets $V=E_f(\sup |f(x)|<1,\ x\varepsilon G)$, where G represents in turn all the bounded sets in E. The space E^{**} conjugate to E^* may now be introduced with its topology. Since $E^{**}\supset E$ there are defined in E the original topology as well as the second conjugate topology, hereinafter referred to as the cc topology. The topology of E is weaker than the cc topology. But ordinary and cc boundedness coincide. If E is of the second category, then its topology coincides with its cc topology, and E^* is complete and weakly complete. Another type of conjugate space E^0 is introduced based on sequentially continuous functionals: $\phi(x)$ is sequentially continuous if $x_n\to x$ implies $\phi(x_n)\to\phi(x)$. Once more the topology in E induced by that in E^{00} is contrasted with the original topology in E. Various examples are given. *E. R. Lorch* (New York, N. Y.).

Michal, Aristotle D. and Wyman, Max. Characterization of complex couple spaces. Ann. of Math. (2) **42**, 247–250 (1941). [MF 3682]

A complex couple space is formed from a real Banach space E by taking pairs $\{f, g\}$, f and g in E, which behave like the symbols $f+ig$. Such a complex space is characterized by the conjugation $\overline{\{f, g\}}=\{f, -g\}$.

F. J. Murray (New York, N. Y.).

ANALYSIS

Theory of Functions of Complex Variables

***Levinson, Norman. Gap and Density Theorems.** American Mathematical Society Colloquium Publications, v. 26. American Mathematical Society, New York, 1940. viii+246 pp. $4.00.

The author has collected in this book his original contributions on a series of related subjects; some of these contributions were published earlier in various periodicals, others are published here for the first time. The book doubtless has a certain unity, but it seems to me difficult, if not impossible, to circumscribe the subject or characterize the methods in simple general terms. Subject and methods alike are greatly influenced by the work of Wiener, and, especially, by Paley and Wiener's book "Fourier Transforms in the Complex Domain" [New York, 1934], of which the present book constitutes, in some respects, a continuation. The combination of "complex variables" and "real variables" in methods and results also recalls the work of Paley and Wiener. A good part of the book is connected with the work of G. Pólya [Math. Z. **29**, 549–640 (1929), and problems 105–108, Jber. Deutsch. Math. Verein. **40**, 80–81 (1931)]. Some of the subjects treated here were considered earlier by V. Bernstein, Miss Cartwright and others. The

last part of the book is devoted to a wide generalization of a theorem of Hardy and Littlewood. The author contributes something essential to all his subjects, obtains very precise results and gives new proofs. Some of his proofs are long, difficult and highly technical, but the details are presented with much care and precision.

The book may be divided into 4 parts. We must confine ourselves to a few sketchy remarks about the contents of each. (1) Which properties of the sequence of functions $\{e^{inx}\}$, $n=0, \pm1, \pm2, \cdots$, are shared by the more general sequence $\{e^{i\lambda_n x}\}$, where λ_n is a sequence of real numbers? Under what conditions is it possible to ensure that a function, orthogonal to each $e^{i\lambda_n x}$, vanishes identically? Under what conditions does there exist an expansion linear in $e^{i\lambda_n x}$ analogous to the Fourier series which is linear in e^{inx}? These are the questions treated in Chapters I, II, IV. Chapter III is concerned with the main tool used in Chapters I and II: with relations between the increase and the zeros of certain entire functions of exponential type. (2) Chapters V–VII are devoted to the following subjects: Fourier transforms which cannot vanish over any interval without vanishing identically; Dirichlet series whose successive exponents have a difference greater than a fixed positive lower bound; analytic functions whose rate of growth along a line is determined by their growth on a sequence of points of that line. The treatment of the last subject proceeds along the same lines as a paper by A. Pfluger [Compositio Math. 4, 367–372 (1936)]. (3) If an entire function of order 1 and type 0 is bounded for all integers, then it is a constant [Pólya]. In Chapters VIII and IX, "best" generalizations of this theorem are given. The main tool is a general lemma on analytic functions. (4) If $\mu_1, \mu_2, \cdots$ are positive, and if there is a θ, $\theta>1$, such that $\mu_{n+1}>\theta\mu_n$ for $n=1, 2, \cdots$, then the convergence of the series $a_1e^{-\mu_1 x}+a_2e^{-\mu_2 x}+\cdots$ for $x>0$ and the existence of its limit for $x\to0$ imply its convergence for $x=0$. This remarkable theorem of Hardy and Littlewood is generalized in Chapters X–XII. Instead of the sequence $\{e^{-\mu_n x}\}$ a general sequence derived from an arbitrary function $K(x)$ of a certain class is considered.

G. Pólya (Providence, R. I.).

Levinson, Norman. An integral inequality of the Phragmén Lindelöf type. J. Math. Phys. Mass. Inst. Tech. 20, 89–98 (1941). [MF 3844]

If $H(z)$ is analytic for $y\geqq0$, if

$$\int_0^\infty |H(\pm x)|^p dx \leqq M$$

for some positive p, and if

$$(*) \qquad \overline{\lim_{r\to\infty}}\, r^{-1}\log|H(re^{i\theta})| \leqq a$$

uniformly in $0\leqq\theta\leqq\pi$, then

$$\int_0^\infty |H(re^{i\theta})e^{-ar\sin\theta}|^p dr \leqq M, \qquad 0\leqq\theta\leqq\pi.$$

[The author does not state that (*) is to hold uniformly, but his proof seems to use the assumption.] A conformal mapping transforms the theorem into an analogous one for an arbitrary angle. The author's proof depends on replacing $H(z)$ by a function $H_1(z)$ which has no zeros for $y>0$, satisfies $|H_1(x)|=|H(x)|$ for real x, and $|H_1(z)|\geqq|H(z)|$ for $y>0$, and has suitable growth properties. [The author's theorem can also be derived by straight-forward Phragmén-Lindelöf arguments from a result of Gabriel [J. London Math. Soc. 2, 112–117 (1927); cf. also Ganapathy Iyer, Quart. J. Math., Oxford Ser. 7, 294–299 (1936)].]

R. P. Boas, Jr. (Durham, N. C.).

Privaloff, I. I. Sur l'intégrale du type de Cauchy-Stieltjes. C. R. (Doklady) Acad. Sci. URSS (N.S.) 27, 195–197 (1940). [MF 3214]

This note is an announcement of results. An "integral of Cauchy-Stieltjes type" is an integral

$$\frac{1}{2\pi i}\int_L \frac{e^{i\varphi}dF(s)}{x-z} = \frac{1}{2\pi i}\int_0^l \frac{e^{i\varphi(s)}dF(s)}{x(s)-z},$$

where L is a closed rectifiable Jordan curve of length l, $F(s)$ is of bounded variation on $(0, l)$, and φ is the angle between the tangent to L and the direction of the positive real axis. When z is on L, the integral is to be taken as a principal value; one of the author's results is that this principal value exists at almost all points of L [cf. R. C. Young and S. Pollard, Proc. London Math. Soc. (2) 28, 293–300 (1929)]. An integral of Cauchy-Stieltjes type is called a Cauchy-Stieltjes integral if its limit as z approaches a point $x(s)$ of L from inside L on non-tangential paths is $F'(s)$ for almost all s. The author states a formula connecting the limit of an integral of Cauchy-Stieltjes type and the corresponding principal value, generalizing his earlier results for absolutely continuous functions $F(s)$ [Privaloff, Saratoff thesis, 1918; see Jahrbuch über die Fortschritte der Mathematik 47, 296–298. Cf. Doklady Akad. Nauk SSSR (N.S.) 23, 859–862 (1939); also these Rev. 1, 305]. Hence follows a necessary and sufficient condition for an integral of Cauchy-Stieltjes type to reduce to a Cauchy-Stieltjes integral, namely that it should vanish identically for z outside L, or equivalently that

$$\int_L e^{i\varphi}x^n dF(s)=0, \qquad n=0, 1, 2, \cdots.$$

In this case the function represented by the Cauchy-Stieltjes integral is also represented by its Cauchy-Lebesgue integral

$$\frac{1}{2\pi i}\int_L \frac{F'(s)dx}{x-z}.$$

The representation of a function by a Cauchy-Stieltjes integral is essentially unique. The author also announces a generalization of the classical result of F. and M. Riesz [Zygmund, Trigonometrical Series, 1935, p. 158] that the boundary function of a function analytic in a circle, if of bounded variation, is absolutely continuous. He replaces the circle by a general rectifiable Jordan curve, and "localizes" the theorem: if the boundary function is of bounded variation on an arc of the boundary curve, it is absolutely continuous on every interior arc. *R. P. Boas, Jr.*

Privaloff, I. Sur l'intégrale du type de Cauchy-Stieltjes. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 4, 261–276 (1940). (Russian. French summary) [MF 3322]

Exposition of the results announced in the paper reviewed above.

Golusin, G. M. On limiting values of Cauchy integral. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. 6, 43–47 (1939). (Russian) [MF 3287]

A simplified proof of a known theorem of Privaloff con-

cerning the limiting values of Cauchy integral extended over a rectifiable Jordan curve. *J. D. Tamarkin.*

Khaplanov, M. G. Sur les coefficients de Taylor d'une classe de fonctions méromorphes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 678–683 (1940). [MF 3613]

If

$$f(z)=\sum_{k=1}^{\infty}\left[\frac{a_k}{(z-\alpha_k)^p}+\frac{b_k}{(z-\alpha_k)^{p-1}}+\cdots+\frac{l_k}{z-\alpha_k}\right.$$

$$\left.+(-1)^{p-1}a_k\left(\frac{1}{\alpha_k^p}+\cdots+\frac{C_{n+p-1}^{p-1}z^{n-1}}{\alpha_k^{n+p-1}}\right)+l_k\left(\frac{1}{\alpha_k}+\cdots+\frac{z^{n-1}}{\alpha_k^n}\right)\right],$$

and if the series

$$\sum_k\frac{|a_k|}{|\alpha_k|^{n+p}},\quad\sum_k\frac{|b_k|}{|\alpha_k|^{n+p-1}},\quad\cdots,\quad\sum_k\frac{|l_k|}{|\alpha_k|^{n+1}}$$

is convergent, then the Taylor's coefficients of $f(z)$ are, for $n>n_0$, of the form $\phi(n)$, where

$$\phi(\zeta)=\omega_p(\zeta)+\omega_{p-1}(\zeta)\zeta+\cdots+\omega_1(\zeta)\zeta^{p-1},$$

the ω being Dirichlet's series with complex exponents. The author proves also a theorem which is almost the converse of the stated theorem. *S. Mandelbrojt* (Houston, Tex.).

Herzog, Fritz. Uniqueness theorems for rational functions. Bull. Amer. Math. Soc. **46**, 942–950 (1940). [MF 3453]

R. Nevanlinna has proved [Le théorème de Picard-Borel et la théorie des fonctions méromorphes, Paris, 1929, p. 109] that two meromorphic functions $f(z)$ and $g(z)$ are identical if for five different W_k the two equations $f(z)=W_k$ and $g(z)=W_k$ have the same finite distinct roots (not counting multiplicities). In the present paper, Herzog proves that, if $f(z)$ and $g(z)$ are rational functions, only four different W_k are required for the above theorem and that, if $f(z)$ and $g(z)$ are polynomials, only two different finite W_k are needed for the theorem. In analogy with Nevanlinna [ibid., pp. 121–128], Herzog also discusses the extent to which just three different W_k determine a rational function $f(z)$; that is, he lists various necessary conditions on rational functions $f(z)$ and $g(z)$ in order that $f(z)\not\equiv g(z)$ when three different W_k are given. The method of proof used in this paper is elementary, consisting largely of comparing the degrees of $f(z)-W_k$, $f'(z)$ and $f(z)-g(z)$, and the total multiplicities of these functions in the roots of the above equations. *M. Marden* (Madison, Wis.).

Kunugui, Kinjiro. Sur un problème de M. A. Beurling. Proc. Imp. Acad. Tokyo **16**, 361–366 (1940). [MF 3478]

Let D be a domain, z_0 a point of its boundary and $f(z)$ a meromorphic function defined in D; $S(D, z_0)$ is defined to be the set of limiting values of $w=f(z)$ as z in D tends to z_0. If $A_n=\sum S(D, z')$, where the summation extends over all boundary points z' of D such that $0<|z_0-z'|<1/n$, and, if $\bar{A}_n$ is the closure of A_n, then $\Pi_1^{\infty}\bar{A}_n$ is denoted by $S(\Gamma, z_0)$. The point z_0 is said to be simple if every neighborhood V of z_0 contains a neighborhood W of z_0 such that any two points of $W\cdot D$ can be joined by a continuous curve in $V\cdot D$.

The conjecture of Beurling is as follows: If $f(z)$ is holomorphic in D, and z_0 is simple and regular with respect to the Dirichlet problem for D, then, for any point w of the set $S(D, z_0)-S(\Gamma, z_0)$, the equation $f(z)-w=0$ has infinitely many roots near z_0, with the possible exception of one value of w. The author weakens this theorem by assuming that $f(z)$ is meromorphic and conjecturing that $f(z)-w=0$ has infinitely many roots near z_0, with the possible exception of two values of w. He then demonstrates by an example that even this weaker theorem, and hence that of Beurling, is not true: his function $f(z)$ is holomorphic and omits in D three values from $S(D, z_0)-S(\Gamma, z_0)$. The author shows next that with a certain modification his revised conjecture is true. The set $S(D, z_0)-S(\Gamma, z_0)$ is open, and so is equal to a sum of connected open sets $\sum_1^{\infty}\Omega_n$. If $f(z)$ is meromorphic, and if z_0 is a non-isolated point of the boundary, nothing being said about its simplicity or regularity, then for each w of Ω_n, $f(z)-w=0$ has infinitely many roots near z_0, with the possible exception of two values of w from Ω_n.

J. W. Green (Rochester, N. Y.).

Morduchaj-Boltovskoi, D. D. On the growth of a transcendental function expressible in finite form in terms of elementary transcendentals. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 172–192. (Russian) [MF 3512]

In this paper the author investigates: (1) the growth of functions expressible in finite form (only functions of the first and second classes are studied in detail); (2) the circumstances under which a function expressible in finite form should be referred to the exponential type. Results obtained for the growth of the function y (expressible in finite form) are used to obtain upper bounds (expressible in finite form) for the coefficients of the series

$$y=a_0+a_1x+\cdots+a_nx^n+\cdots.$$

D. C. Spencer (Cambridge, Mass.).

Kamenetzky, I. M. Sur l'indicatrice de la croissance d'une fonction entière du premier ordre et sur la distribution des singularités d'une fonction représentée par la série associée de Taylor. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 552–555 (1940). [MF 3565]

The author proves that a necessary and sufficient condition for ϕ to be the direction of maximal increase for the entire function $f(z)=\sum a_nz^n/n!$ of the first order and of the normal type A is that $F(\lambda-\phi)=A+\lambda$, where

$$F(\lambda,\alpha)=\overline{\lim_{n=\infty}}\left(\left|\sum_{k=0}^{n}C_n^kt^{n-k}a_k\right|\right)^{1/n},\quad t=\lambda e^{i\alpha};\ \lambda>0.$$

S. Mandelbrojt (Houston, Tex.).

Lévine, B. Sur certaines applications de la série d'interpolation de Lagrange dans la théorie des fonctions entières. Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 437–454 (1940). (Russian. French summary) [MF 3741]

The author considers interpolation of entire functions over rather general sets of points in the complex plane with considerable success. The points of interpolation $\{a_k\}$ satisfy for some $\rho>0$

$$\lim_{r=\infty}n(r,\psi)r^{-\rho}=\Delta_\rho(\psi),\qquad|\psi|\leqq\pi,$$

where $n(r, \psi)$ is the number of $\{a_k\}$ in the sector $|z|\leqq r$, $0<\arg z<\psi$. In addition there is a spacing requirement on $\{a_k\}$. If ρ is an integer it is assumed that $a_ke^{\pi im/\rho}$ $(m=1, 2, \cdots, \rho-1)$ are contained in $\{a_k\}$. Entire functions $F(z)$ of order ρ are then set up with $\{a_k\}$ as zeros and their growth functions denoted by $H(\theta)$. The author then shows that, if $G(z)$ is an entire function of order ρ and growth

function $h(\theta)$ and if $h(\theta)<H(\theta)$, then

$$G(z)=F(z)\sum_1^\infty \frac{G(a_k)}{F'(a_k)(z-a_k)}.$$

The author extends considerably a result of Ganapathy Iyer and the reviewer and obtains results on the growth of an entire function at infinity on the basis of its growth on $\{a_k\}$.

N. Levinson (Cambridge, Mass.).

Cartwright, M. L. On level curves of integral functions. Quart. J. Math., Oxford Ser. **11**, 277–290 (1940). [MF 3790]

A curve on which $|f(z)|$ is constant is called a level curve. Level curves which extend to infinity are called open level curves and it is with these that this paper is concerned. For closed level curves it was shown by Valiron that if $f(z)$ and $g(z)$ are integral functions both of which have a simple closed curve c as a level curve, then either

$$f(z)=K\{g(z)\}^\alpha$$

for real α or else there exists a number w_0 and an integral function $w(z)$ such that $|w(z)|=1$ on c and $w(z)$ does not take the value w_0. In the latter case any integral function $f(z)$ for which c is a level curve can be expressed in the form

$$f(z)=K\{w(z)\}^n\left\{\frac{w(z)w_0-1}{w(z)-w_0}\right\}^m,$$

where m and n are positive integers or zero. The case of open level curves is not so simple. The author obtains many results including the following: If $f(z)$ and $g(z)$ are integral functions of finite order having the complete level curve c on which $|f|=M$ and $|g|=M'$ in common, then the above stated conclusion of Valiron's theorem holds and moreover either m or n must be zero. *N. Levinson.*

Bohr, Harald. Zum Picardschen Satz. Mat. Tidsskr. B. **1940**, 1–6 (1940). [MF 3876]

Bohr, Harald. Zum Picardschen Satz. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 29–33. [MF 3499]

[The two papers are identical.] The author proves the following extension of Picard's (second) theorem: Let S be the Riemann surface of the logarithm ($z=re^{i\theta}$, $r>0$, $-\infty<\theta<\infty$). Suppose that the function $f(z)$ is regular in the sub-region $|z|>R$, $-\infty<\theta<\infty$, of S and different from a and b (a and b are complex numbers, $a\neq b$). Moreover, suppose that for $|z|=\rho>R$, $-\infty<\theta<\infty$, $|f(z)|<k$ (k is a constant). Then there exists a positive K which depends on R, ρ, a, b and k only such that for all sufficiently large $|z|$, uniformly for $-\infty<\theta<\infty$, $|f(z)|<|z|^K$. Picard's theorem is obviously a special case of this result (when $f(z)$ is single-valued for $|z|>R$). This generalization is obtained from the Schottky-Landau theorem [as stated in Nachr. Ges. Wiss. Göttingen **1910**, § 6] by a method which is an extension of Lindelöf's derivation of Picard's theorem from Schottky's theorem. The author remarks that his method may also be applied to obtain his generalization of Picard's theorem for almost periodic analytic functions which was given by him in Acta Math. **47**, 270 (1926) and thus to simplify his original proof of it. *S. E. Warschawski.*

Biggeri, Carlos. On Picard's second theorem. Bol. Mat. **13**, 291–294 (1940). (Spanish) [MF 3444]

A slight modification in the passage from Bloch's theorem to Picard's theorem. *S. Mandelbrojt* (Houston, Tex.).

Dassen, C. C. Concerning a proof of Picard's second theorem. Bol. Mat. **13**, 256–259 (1940). (Spanish) [MF 3347]

The author reproduces part of a letter from C. Biggeri, containing a proof of an elementary lemma from the theory of functions; this lemma is relevant to the proof mentioned in the title [evidently that given by Blaquier in the paper reviewed below]. *R. P. Boas, Jr.* (Durham, N. C.).

Blaquier, Juan. A proof of the two famous theorems of Picard. An. Soc. Ci. Argentina **129**, 145–152 (1940). (Spanish) [MF 3775]

The first Picard theorem states that an entire function which omits two values is a constant; the second, that an analytic function cannot omit two values in every neighborhood of an essential singular point. To establish the first theorem by using Bloch's theorem, one constructs a function $g(z)$ such that, if $f(z)$ omits two values, $h(z)=g[f(z)]$ is entire and omits a set of values whose intersection with every sufficiently large circle is not empty; it then follows from Bloch's theorem that $h(z)$ is constant and therefore that $f(z)$ is constant. To establish the second theorem, one ordinarily appeals to Schottky's theorem. The author attempts to give a simpler proof of the second Picard theorem by a method analogous to that outlined here for the first. Assuming that $f(z)$ is analytic for $|z|>R$, and that $f(z)$ has an essential singularity at $z=\infty$ but omits two values, he constructs a function $h(z)$ omitting so many values that a contradiction results from Bloch's theorem. However, it is not clear that the function $h(z)$ constructed is single-valued [cf. the following review], and so the proof is incomplete. *R. P. Boas, Jr.* (Durham, N. C.).

Dassen, C. C. Second theorem of Picard. A demonstration due to Doctor Carlos Biggeri. An. Soc. Ci. Argentina **131**, 4–15 (1941). (Spanish) [MF 3897]

A discussion of proofs of Picard's two theorems based on Bloch's theorem, with particular attention to the difficulty about the function $h(z)$ noted in the preceding review. *R. P. Boas, Jr.* (Durham, N. C.).

Shah, S. M. A note on maximum modulus and the zeros of an integral function. Bull. Amer. Math. Soc. **46**, 909–912 (1940). [MF 3447]

If $f(x)$ is an entire function, $n(r)$ denotes the number of its zeros in $|z|\leqq r$, and $T(r)$ is its Nevanlinna characteristic function. For any $\rho\geqq 0$ and $\psi(x)>0$ such that

$$\limsup_{x\to\infty}\frac{\log\psi(x)}{\log x}\leqq\rho,$$

the author constructs an entire function $f(z)$ of order ρ for which $\limsup T(r)/\{\psi(r)n(r)\}=\infty$, and another for which $\liminf \psi(r)T(r)/n(r)=0$. *R. P. Boas, Jr.*

Ganapathy Iyer, V. On maximal integral functions. J. Indian Math. Soc. (N.S.) **4**, 97–115 (1940). [MF 3272]

Let $g(z)$ be an entire function and ρ and d two positive numbers. If

$$\lim\frac{\log|g(z)|}{|z|^\rho}=d$$

as $|z|\to\infty$ outside a system of circles the sum of whose radii is finite, then $g(z)$ is said to belong to $M(\rho, d)$. The author studies this class of functions as well as a certain subclass. An example of the theorems proved is that, if

$n(r, \alpha, \beta)$ are the number of zeros of a function of class $M(\rho, d)$ in the sector $|z| \leqq r$, $\alpha \leqq \theta \leqq \beta$, then

$$\lim_{r\to\infty} \frac{n(r, \alpha, \beta)}{r^\rho} = \frac{d\rho}{2\pi}(\beta-\alpha).$$

N. Levinson (Cambridge, Mass.).

Hille, Einar. A class of differential operators of infinite order, I. Duke Math. J. 7, 458–495 (1940). [MF 3414]

Extending earlier investigations [Hille, Duke Math. J. 5, 875–936 (1939); cf. these Rev. 1, 141] the author develops a comprehensive theory of the differential operator $G(\delta_s) \equiv \sum_{k=0}^{\infty} g_k \delta_s^k$, where $\delta_s = z^2 - d^2/dz^2$ is the Hermite-Weber operator, $\delta_s^k = \delta_s \cdot \delta_s^{k-1}$, and $G(w) = \sum_{k=0}^{\infty} g_k w^k$ is an entire function whose precise characterization is to be determined by the class of functions $\{f(z)\}$ to which $G(\delta_s)$ is to be applicable (that is, such that $G(\delta_s)\cdot f(z)$ is to converge at every point where $f(z)$ is holomorphic). In §§ 1–2, $\delta_s^k f(z)$ and $G(\delta_s)\cdot f(z)$ are expressed in terms of Cauchy integrals, and estimates derived for $\delta_s^k f(z)$, resulting in upper bounds for

$$\limsup_{k\to\infty} k^{-\lambda} |\delta_s^k f(z)|^{1/k},$$

where $f(z)$ is entire, of order ρ and type α, in the following cases: $\rho > 2$, $\lambda = 1/\sigma$, σ = order conjugate to ρ, defined by $(1/\rho)+(1/2\sigma)=1$; $\rho=2$, $\lambda=1$; $\rho<2$, $\lambda=1$, or $\rho=2$, $\alpha=0$, $\lambda=1$.

In § 3 the functional

$$F_\lambda(z; f) \equiv \limsup_{k\to\infty} \left[\frac{|\delta_s^k f(z)|}{\Gamma(1+k/\lambda)}\right]^{1/(2k)}$$

is examined. (It plays a rôle in the $G(\delta_s)$-theory analogous to that of the functional

$$\Phi_\sigma(z; f) \equiv \limsup_{k\to\infty} \left|\frac{f^{(k)}(z)}{\Gamma(1+k/\sigma)}\right|^{1/k}$$

in the theory of the operator $G(d/d_z)$.) If $\mathfrak{H}_{\rho,\alpha}$ is the class of entire functions $f(z)$ for which

$$\limsup_{r\to\infty} r^{-\rho} \log M(r; f) \leqq \alpha,$$

$M(r; f)$ having its customary meaning, then, for $f \in \mathfrak{H}_{\rho,\alpha}$, $F_\lambda(z; f) \equiv 0$ for $\rho > 2$, $\lambda < \sigma$ (= conjugate order to ρ); and for $\rho \leqq 2$, $\lambda < 1$; $F_\sigma(z; f) \leqq (\alpha\rho)^{1/\rho}\cdot(2\sigma)^{1/(2\sigma)}$ for $\rho > 2$. (Best estimate of its kind.) There are further results on the boundedness and continuity of the functional F.

Let $\mathfrak{G}_{\sigma,\beta}$ be the class of functions $G(w)$ for which

$$\limsup_{r\to\infty} r^{-\sigma} \log \max_{0\leq\theta<2\pi} |G(re^{i\theta})| \leqq \beta.$$

It is shown in § 4 that $G(\delta_s)$ is applicable (as earlier defined) to all analytic functions if and only if $G(w) \in \mathfrak{G}_{1,0}$. Also a mapping theorem: If $D_1 \supset D_2$ are bounded domains, D_2 being at a positive distance from the complement of D_1; if $\mathfrak{H}(D_i)$, $i=1, 2$, is the class of functions holomorphic and bounded in D_i; and if $G(w) \in \mathfrak{G}_{1,0}$, then $G(\delta_s)$ defines a linear, bounded continuous transformation on $\mathfrak{H}(D_1)$ to $\mathfrak{H}(D_2)$. A further result is that, if the sequence $\{\Gamma_n(w)\} \in \mathfrak{G}_{1,0}$ converges to $G(w)$, and if $\Gamma_0(w) \in \mathfrak{G}_{1,0}$ exists such that $M(r, \Gamma_n) \leqq M(r, \Gamma_0)$ for all n and r, then, for every function $f(z)$ holomorphic in a domain D, $P_n(\delta_s)\cdot f(z)$ converges to $G(\delta_s)\cdot f(z)$ in D, uniformly in every bounded domain $D_0 \subset D$ lying at a positive distance from the complement of D. It is also shown that $G(\delta_s)$ applies to all entire functions if and only if $G(w) \in \mathfrak{G}_{1,\beta}$ for some (finite) β.

Given α, the conjugate type β is defined by

$$(\alpha\rho)^{1/\rho}\cdot(2\beta\sigma)^{1/(2\sigma)} = 1.$$

(σ is the order conjugate to ρ, as before, with the extra definition: $\sigma = 1$ for $\rho < 2$.) §§ 5–6 give further applicability theorems: In order that $G(\delta_s)$ be applicable to all classes $\mathfrak{H}_{\rho,\alpha}$ for a fixed $\rho \geqq 2$ ($0 \leqq \alpha < \infty$) it is necessary and sufficient that $G(w) \in \mathfrak{G}_{\sigma,0}$. If $\rho < 2$ the condition is sufficient; and an analogous result with ρ and α both fixed. Given the order and type of $f(z)$ and $G(w)$, upper bounds (best of their kind) are found in § 7 for the order and type of $G(\delta_s)\cdot f(z)$. In the last section are gathered together various examples that serve to complete the proofs of some of the earlier theorems. Also, the functionals $F_\lambda(z; f)$ and $\Phi_\sigma(z; f)$ (already referred to) are compared and contrasted. *I. M. Sheffer.*

Wolff, J. Sur les fonctions holomorphes dont l'ensemble des valeurs est soumis à certaines restrictions. Nederl. Akad. Wetensch., Proc. 43, 1018–1022 (1940). [MF 3533]

The author states and proves inequalities for the angular derivative (Carathéodory's Winkelderivierte) of functions which satisfy the hypothesis of Schwarz's lemma in $|z| < 1$, analogous to those given by H. Unkelbach [Math. Ann. 115, 203–236 (1938)]. He proves among other things the following result. Let $w(z)$ be regular and $|w(z)| < 1$ in $|z| < 1$, and let $c(p, q)$ be a circular arc in $|w| < 1$ of which the extremities p and q lie on the unit circumference $r(|w| = 1)$. If $w(z)$ omits some value w on the open arc $c(p, q)$, and if the sectorial (angular) limits of $w(z)$ are p and q at p and q, respectively, then the angular derivatives $w'(p)$ and $w'(q)$ satisfy the inequality

$$w'(p)w'(q) \geqq \psi^2/\sin^2\psi,$$

where ψ is the angle between $c(p, q)$ and r.

D. C. Spencer (Cambridge, Mass.).

Wall, H. S. A class of functions bounded in the unit circle. Duke Math. J. 7, 146–153 (1940). [MF 3395]

The functions considered are the so-called moment generating functions

$$(*) \qquad f(x) = \int_0^1 \frac{d\varphi(u)}{1+xu}$$

($\varphi(u)$ bounded and non-decreasing) discussed by the author [Trans. Amer. Math. Soc. 48, 165–184 (1940); these Rev. 2, 90]. Functions (*) can be represented by continued fractions [Wall, op. cit.]; inequalities for such continued fractions are derived from work of Scott and Wall [Trans. Amer. Math. Soc. 47, 155–172 (1940); these Rev. 1, 217]; and certain transformations of continued fractions considered by Garabedian and Wall [Trans. Amer. Math. Soc. 48, 185–207 (1940); these Rev. 2, 90] and Wall [op. cit.] are reflected in corresponding transformations of the functions, hence in further inequalities. The general inequalities are applied to the hypergeometric function $F(\alpha, 1, \gamma; -x)$ ($0 < \alpha < \gamma$), and to some more elementary functions. A typical inequality is

$$\left|f(z) - \frac{1}{2-g_1}\right| \leqq \frac{1-g_1}{2-g_1}, \qquad |z| \leqq 1,$$

where $f(z)$ is a function (*) such that $|f(z)| \leqq 1$ for $|z| \leqq 1$, and $g_1 = f(0)$. Inequalities involving $\log(1+z)$, recently set by the author as a problem [Amer. Math. Monthly 47, 491 (1940)], are special cases of results from the present paper.

R. P. Boas, Jr. (Durham, N. C.).

Carlson, Fritz. Sur les coefficients d'une fonction bornée dans le cercle unité. Ark. Mat. Astr. Fys. **27A**, no. 1, 8 pp. (1940). [MF 3493]

Let E denote the class of functions

$$f(z)=a_0+a_1z+a_2z^2+\cdots, \qquad |a_n|=\alpha_n,$$

which are regular and of modulus not greater than 1 for $|z|<1$. The necessary and sufficient conditions upon the coefficients a_n in order that $f(z)$ belong to E are well known and due to J. Schur [J. Reine Angew. Math. **147** (1917), **148** (1918)]. The author obtains further inequalities for the coefficients. For example,

$$\text{(A)} \qquad \alpha_{2n+1}\leqq 1-\alpha_0^2-\alpha_1^2-\cdots-\alpha_n^2, \quad n=0, 1, \cdots,$$

$$\text{(B)} \qquad \alpha_{2n}\leqq 1-\alpha_0^2-\alpha_1^2-\cdots-\alpha_{n-1}^2-\alpha_n^2/(1+\alpha_0), \quad n=1, 2, \cdots.$$

In order that the equality signs hold in (A) it is necessary and sufficient that $f(z)$ be a rational function of the form

$$f(z)=\frac{a_0+a_1z+\cdots+a_nz^n+\epsilon z^{2n+1}}{1+(\bar{a}_1z^n+\cdots+\bar{a}_0z^{2n+1})\epsilon}, \qquad |\epsilon|=1,$$

having modulus 1 on $|z|=1$. For equality in (B) we have $f(z)$ of the form

$$f(z)=\frac{a_0+a_1z+\cdots+a_nz^n/(1+\alpha_0)+\epsilon z^{2n}}{1+(\bar{a}_nz^n/(1+\alpha_0)+\cdots+\bar{a}_0z^{2n})\epsilon}, \qquad |\epsilon|=1,$$

where $a_0\bar{a}_n^2\epsilon$ is real and non-positive. The method involved permits one to obtain several other inequalities of an analogous character, some of which are given at the end of this paper. *M. S. Robertson* (New Brunswick, N. J.).

Kryloff, W. I. Some simple remarks on classes of analytic functions in a circle. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **6**, 71–80 (1939). (Russian) [MF 3291]

Let $f(z)$ be analytic in the circle $|z|<1$. R. Nevanlinna [Acta. Soc. Sci. Fennicae **4**, 1–40 (1922)] introduced the class N of functions $f(z)$ which can be represented as a ratio of two functions which are analytic and bounded in $|z|<1$. The class N is also characterized by the condition

$$\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|\,d\theta\leqq M_f<\infty, \qquad 0\leqq r<1.$$

The subclass of N characterized by the condition

$$\lim_{r\to 1}\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|\,d\theta=\int_{-\pi}^{\pi}\log^+|f(e^{i\theta})|\,d\theta$$

is denoted by S_p. This class of functions was investigated by W. Smirnoff [Bull. Acad. Sci. URSS [Izvestia Akad. Nauk SSSR] (7) **1932**, 337–371]. In the present paper the author considers two other subclasses of N, characterized respectively by

$$\lim_{r\to 1}\int_{-\pi}^{\pi}\log^-|f(re^{i\theta})|\,d\theta=\int_{-\pi}^{\pi}\log^-|f(e^{i\theta})|\,d\theta,$$

$$\lim_{r\to 1}\int_{-\pi}^{\pi}\big|\log|f(re^{i\theta})|\big|\,d\theta=\int_{-\pi}^{\pi}\big|\log|f(e^{i\theta})|\big|\,d\theta,$$

and establishes for these two classes of functions properties analogous to those proved for S_p by Smirnoff.

J. D. Tamarkin (Providence, R. I.).

Gelfer, S. Zur Theorie der multivalenten Funktionen. Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 239–250 (1940). (Russian. German summary) [MF 3461]

Let S_p be the class of functions

$$w=f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n$$

which are regular and p-valent in $|z|<1$, and let $d=\inf_{|z|<1}|f(z)/z^p|$. The author proves for the class S_p that: (i) $d>\{4/27\cdot\ln 3\}^p$;

(ii) $$|f(z)|>\{16/27\cdot\ln 3\}^pr^p/(1+r)^{2p}, \quad \text{for } |z|=r<1.$$

The result (i) is obtained by subordination arguments (Lindelöf's principle), and (ii) is deduced from (i) by a well-known device. In the second half of the paper the author considers the class E_ρ of functions

$$f(z)=z^k+\sum_{n=k+1}^{\infty}a_nz^n, \qquad k\geqq 1,$$

which are regular in $|z|<1$, and schlicht in any circle of fixed radius ρ, $\rho\leqq 1$, which touches the circumference $|z|=1$ from the inside. He proves among other things: (i) if $f\in E_\rho$, $\rho>\frac{1}{2}$, then

$$(2\rho-1)^3(1-r)/(2\rho-1+r)^2\leqq|f'(z)|\leqq(2\rho-1+r)/(2\rho-1)(1-r)^2,$$
$$|\arg f'(z)|\leqq 2\ln(2\rho-1+r)/(2\rho-1)(1-r),$$
$$(2\rho-1)^2r/(1+r)^2\leqq|f(z)|\leqq\rho r/(2\rho-1)(1-r)^2,$$

for $|z|=r<1$, and $|a_n|<\rho en^2/(2\rho-1)$; (ii) if $f\in E_\rho$, $\rho\leqq\frac{1}{2}$, then $|f(z)|<B(k,\rho)\cdot r/(1-r)^2$ on $|z|=r<1$, and $|a_n|<B(k,\rho)en^2$; and (iii) if $f\in E_\rho$, and if f is p-valent, then $|a_n|<C(k,\rho)pn$.

The reviewer would like to point out that a lower bound for d larger than that given above does in fact exist. The larger constant is an immediate consequence of a lower bound for d for mean 1-valent functions obtained by the reviewer in a paper to appear shortly in the Annals of Mathematics. *D. C. Spencer* (Cambridge, Mass.).

Golusin, G. M. Über p-valente Funktionen. Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 277–284 (1940). (Russian. German summary) [MF 3464]

The author considers the following two classes of p-valent functions: (i) the class S_p of functions $w=f(z)=z^p(1+a_1z+a_2z^2+\cdots)$ which are p-valent and regular in $|z|<1$; (ii) the class Σ_p of functions $w=F(\zeta)=\zeta^p(1+\alpha_1/\zeta+\alpha_2/\zeta^2+\cdots)$ which are p-valent and regular in $|\zeta|>1$ (with exception of a pole at $\zeta=\infty$). He proves among other things the following three results: (1) if $F(\zeta)\in\Sigma_p$, $F(\zeta)\neq 0$ for $|\zeta|>1$, and if $\alpha_1=\alpha_2=\cdots=\alpha_{k-1}=0$ ($k\geqq 1$), then, for $n=k$, $k+1, \cdots, 2k-1$, we have $|\alpha_n|\leqq 2p/n$ with equality only if $F(\zeta)=\zeta^p(1+\eta/\zeta^n)^{2p/n}$, $|\eta|=1$; (2) if $f(z)\in S_p$, and if $a_1=a_2=\cdots=a_{k-1}=0$ ($k\geqq 1$), then, for $n=k, k+1, \cdots, 2k-1$, we have $|a_n|\leqq 2p/n$, with equality only if $F(z)=z^p/(1+\eta z^n)^{2p/n}$, $|\eta|=1$; and (3) if R is the Riemann surface on which f of class S_p maps the circle $|z|<1$, then R covers the circle $w\leqq 1/2^{p+1}$.

The reviewer remarks that results (1) and (2) (which are equivalent) are known for the class of mean p-valent functions (which includes p-valent functions). In fact, the mean p-valency (in particular the p-valency) of $f(z)=z^p(1+a_1z+\cdots)$ implies the mean 1-valency of the function $f_1(z)=\{f(z)\}^{1/p}$, and so results for mean 1-valent functions at once yield corresponding results for the class S_p. For example, the result $|a_2|\leqq 2|a_1|$ for the family of mean 1-valent functions [see the paper reviewed below and

abstract 126, Bull. Amer. Math. Soc. **46**, 55 (1940)] is equivalent to the result $|a_1| \leqq 2p$ for mean p-valent functions of the form $f(z)=z^p(1+a_1z+\cdots)$.

D. C. Spencer (Cambridge, Mass.).

Spencer, D. C. On distortion in analytic transformations. J. Math. Phys. Mass. Inst. Tech. **20**, 124–126 (1941). [MF 3848]

Let $f(z)=a_1z+a_2z^2+\cdots$ be regular in $|z|<1$ and map the unit circle $|z|<1$ on the Riemann domain W. If $n(w)\leqq 1$, where $n(w)$ denotes the number of times W covers the point w, and if d is the distance of the boundary of W from $w=0$, it is well known that $d \geqq |a_1|/4$. The author announces the more general theorem that, if $n(0)=1$, and if there is a real number β for which $n(de^{i\beta})=0$, where

$$d=\inf_{|z|<1}|f(z)/z|,$$

then $d>|a_1|/13$. The theorem is false if $n(de^{i\beta})>0$ for all β. The problem of the best constant involved remains open. In particular the theorem is correct for mean 1-valency functions but with the constant 13 replaced by a much smaller one. If in addition $|f(z)|<M$ for $|z|<1$, then for $0<\rho<1$ the author also finds

$$|\arg f(\rho e^{i\phi})| \leqq B \log 1/(1-\rho)+O(1),$$

where $B=(1/\pi)\log 13M/|a_1|$. Detailed proofs are to appear shortly. *M. S. Robertson* (New Brunswick, N. J.).

Spencer, D. C. On a theorem of Rengels. J. Math. Phys. Mass. Inst. Tech. **20**, 118–123 (1941). [MF 3847]

Let $\phi(r)$ be the annulus $0\leqq r\leqq|z|\leqq 1$, $\phi(0)=\phi$, slit along concentric circular arcs with $z=0$ as center; $f(z)$ with $f(0)=0$, $|f'(0)|=1$ maps ϕ on a Riemann domain W. The maximum distance of the boundary of W from $w=0$ is denoted by D. E. Rengel [Schr. Math. Inst. u. Inst. Angew. Math. Univ. Berlin **1**, 141–162 (1933)] has shown that if W is schlicht then $D>1$. The author generalizes this theorem. Let $\sigma_1, \sigma_2, \cdots$ be the boundary elements of W corresponding to the concentric circular slits of ϕ. Let B denote the remainder of the boundary, U the domain bounded by B, U^* the portion of U seen from $f(0)$, B^* the boundary of U^*, and W^* the sub-domain of W every point of which can be seen either from $f(0)$ or from a point of B^* looking toward $f(0)$ if any one σ_i is opaque. A Riemann domain G is defined to be mean 1-valent (with respect to the origin) if the area (multiply covered points being counted multiply) of that part of G lying in any circle with center 0 does not exceed the area of the circle. With this notation the author proves that, if W^* is mean 1-valent, then $D>1$ unless $f=e^{i\alpha}z$. This theorem contains a result due to A. Bermant [C. R. (Doklady) Acad. Sci. URSS **18**, 137–140 (1938), and C. R. Acad. Sci. Paris **207**, 31–33 (1938)], for the case when there are no slits and W is simply connected, W^* being in this case schlicht. [In the title Rengel is misprinted as Rengels.] *M. S. Robertson* (New Brunswick, N. J.).

Küstner, Herbert. Zur Kontinuitätsmethode in der Theorie der konformen Abbildung. Ber. Verh. Sächs. Akad. Wiss. Leipzig **92**, 51–72 (1940). [MF 3490]

Let $a_1, a_2, \cdots, a_n$ be $n(\geqq 4)$ distinct points on the circumference $|z|=1$ in the z-plane. Then there exist an arcs-polygon Π (terminology suggested by S. Bochner for "Spitzen-polygon") in the Z-plane and a (1, 1) conformal map $Z=\varphi(z)$ of $|z|<1$ onto the interior of Π such that under the continuous extension of $\varphi(z)$ to $|z|=1$ the points $a_1, a_2, \cdots, a_n$ correspond to the vertices of Π. The author establishes this theorem by means of the method of continuity, carrying out in detail the program suggested by Koebe [see, e.g., Koebe, Deutsche Math. **1**, 859–879 (1936)].

M. H. Heins (Princeton, N. J.).

Wolff, J. Théorème sur l'itération d'une représentation conforme. Nederl. Akad. Wetensch., Proc. **43**, 1016–1017 (1940). [MF 3532]

Suppose that $z_1(z)$, $z_1(1)=1$, $z_1(\infty)=\infty$, maps the half-plane $D(\Re z>0)$ conformally on a domain D_1 interior to D, the frontier f of D_1 being a Jordan curve interior to D extending to infinity in the two directions $\arg z=\pm\pi/2$ in such a way that $|z|$ is ultimately increasing on it. Let $z_n(z)=z_{n-1}\{z_1(z)\}$ and $D_n=z_n(D)$ for $n\geqq 2$, and let $N=\lim D_n$ as $n\to\infty$. Under these hypotheses the derivative at infinity, $\lim\{z_1(z)/z\}=\lambda$, say, is independent of the manner in which $z\to\infty$. In an earlier note [C. R. Acad. Sci. Paris **210**, 658–659 (1940); these Rev. **2**, 83] the author has shown that, if $\lambda>0$, N contains a domain of which the angular width at ∞ is π. Here he proves that if N contains a domain Δ containing a point $\alpha\neq 1$ and its image $\alpha_1=z_1(\alpha)$, then $\lambda>0$. *D. C. Spencer* (Cambridge, Mass.).

Khajalia, G. J. Sur la théorie de la représentation conforme des domaines doublement connexes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 550–551 (1940). [MF 3564]

A summary of the author's paper in Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 97–106 (1940); see these Rev. **2**, 84.

D. C. Spencer (Cambridge, Mass.).

Shiffman, M. Uniqueness theorems for conformal mapping of multiply connected domains. Proc. Nat. Acad. Sci. U. S. A. **27**, 137–139 (1941). [MF 3810]

In a previous paper [Courant, Manel and Shiffman, Proc. Nat. Acad. Sci. U. S. A. **26**, 503–507 (1940); these Rev. **2**, 84] it was proved that any k-fold connected domain could be mapped conformally on the exterior of k non-intersecting simple closed curves $\Gamma_1, \cdots, \Gamma_k$ which are respectively homothetic to k given convex analytic curves $C_1, \cdots, C_k$. If $k\geqq 3$, it was further possible to restrict the mapping domains by requiring that Γ_1, Γ_2 and Γ_3 be obtained from C_1, C_2 and C_3 by radial transformations relative to a fixed interior point of each of the C_i, $i=1, 2, 3$; such mapping domains were said to be normalized. The author proves that any k-fold connected domain can be mapped conformally onto a unique normalized domain of the above type (the curves $C_1, \cdots, C_k$ being supposed given). *C. B. Morrey.*

Heins, Maurice H. A note on a theorem of Radó concerning the $(1, m)$ conformal maps of a multiply-connected region into itself. Bull. Amer. Math. Soc. **47**, 128–130 (1941). [MF 3826]

If $w=f(z)$ is an analytic and single valued function defined in a region G_z of the z-plane, such that $z\subset G$ implies $w\subset G$, and such that every value w in G has exactly m antecedents, then $f(z)$ is said to define a $(1, m)$ conformal map of G_z on itself. The author gives a direct proof, without resort to the mapping of multiply connected regions on canonical regions, of a theorem of Radó, which asserts that, if G_z is a region of finite connectivity $p>1$, then there exists no $(1, m)$ conformal map of G_z on itself for $m>1$. By iterating the function $f(z)$ a suitable number of times, the author obtains a function $F(z)$ which maps the various components of the boundary of G_z on themselves. By using

the Nevanlinna principle of harmonic measure, he is able to show that the sequence of successive iterates of $F(z)$ does not tend to a constant, and the assumption that $m>1$ leads to a contradiction. *J. W. Green* (Rochester, N. Y.).

Teichmüller, Oswald. Extremale quasikonforme Abbildungen und quadratische Differentiale. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1939**, no. 22, 197 pp. (1940). [MF 3928]

The author states at the beginning of this paper that while his principal results are not proved, he hopes to give sufficient heuristic justification to remove any serious doubts, and to direct future attempts at proving the results. He deals with a class of generalized finite Riemann surfaces (possibly with boundary): that is, finite connected topological spaces, such that to every neighborhood there belongs a class of topological mappings of the neighborhood onto a region of a complex z-plane, with the property that any two mappings of the common part of two neighborhoods are related to each other by a conformal transformation—this the author allows to be directly or indirectly conformal, so that non-orientable surfaces are taken into consideration. A principal region (Hauptbereich) is such a Riemann surface in which a finite number of points (some possibly on the boundary) are specified; the author is interested in mappings of one principal region onto another, carrying special points into special points. A number of topological properties of the space obtained from a class of homeomorphic principal regions by identifying those which are conformally equivalent are assumed without proof, but verified in several special cases, and used throughout the paper. A quasiconformal map is one for which a certain point-function, called the dilation quotient D, is bounded; for a conformal map, $D\equiv 1$. [Cf. Nevanlinna, Eindeutige analytische Funktionen, Berlin, 1936, p. 344.] The author sets the problem of determining the class of extremal quasiconformal maps of one principal region onto another, that is, the class of maps for which the maximum of D, taken over the region, is a minimum. He assumes without proof, but with suggestive heuristic reasoning, that D is constant for an extremal quasiconformal map. A vector field can be assigned in a natural way to every quasiconformal map, and the author asks which vector fields correspond to extremal quasiconformal maps. He proposes to answer his question by means of quadratic differentials. A differential of dimension n has the form $d\zeta^n=g(t)dt^n$, where $g(t)$ is meromorphic and transforms under change of the local parameter t so that $d\zeta^n$ is invariant. If $n=1$, one has the usual notion of a differential on a Riemann surface; if $n=2$, $d\zeta^n$ is called a quadratic differential.

If a principal region $\mathfrak{H}$ is mapped onto another one $\mathfrak{H}'$, the corresponding local parameters z and z' transform according to a law of the form $|dz'|^2=Edx^2+2Fdxdy+Gdy^2$, and so $ds^2=\lambda(z)|dz'|^2$ ($\lambda(z)$ is a suitable positive function) defines on $\mathfrak{H}$ a new metric (in the sense of differential geometry), and there is a one-to-one correspondence between principal regions $\mathfrak{H}'$ and metrics ds^2. A quasiconformal map of $\mathfrak{H}$ onto $\mathfrak{H}'$ can be defined in terms of the metric (as in the work of Ahlfors), and so the author's problem can be formulated in terms of change of metrics on $\mathfrak{H}$. The use of a metric also allows him to discuss infinitesimal quasiconformal mappings, which he thinks of as determined by an infinitesimal change in the metric, inducing a mapping of one principal region onto another differing infinitesimally from it. He states that it is not possible to make this rather nebulous concept precise by means of limiting processes. An infinitesimal mapping is heuristically characterized by means of a differential invariant, in terms of which the author formulates the problem of characterizing the extremal infinitesimal conformal mappings. Heuristic reasoning leads him to a conjectural solution in terms of quadratic differentials; infinitesimal mappings of the form conjectured are shown to be extremal, but the converse remains a conjecture. The form of the solution for infinitesimal mappings then leads the author to conjecture that the extremal (finite) quasiconformal mappings are characterized by metrics of the form $ds^2=\lambda(|d\zeta|^2+c\Re d\zeta^2)$, where $d\zeta^2$ is a quadratic differential, λ is positive and c is a positive constant (<1). It follows (still as a conjecture) that extremal finite quasiconformal mappings are characterized by the same vector fields obtained for extremal infinitesimal mappings, together with constant dilation quotients (>1); it is shown that mappings with these characteristics are actually extremal quasiconformal.

A number of more special problems are discussed. The results and conjectures are illustrated extensively throughout by study of numerous special cases where they can actually be shown to be true. The author concludes the paper by discussing the significance of the study of quasiconformal mapping for the theory of conformal mapping. *R. P. Boas, Jr.* (Durham, N. C.).

Aronszajn, N. Approximation des fonctions harmoniques et quelques problèmes de transformation conforme. Bull. Soc. Math. France **67**, 137–161 (1939). [MF 2585]

The present article is the first of three parts of a memoir, these parts being entitled: (1) approximations to harmonic functions in the mean; (2) conformal transformations onto star-shaped regions; (3) conformal type and order on curves with sectionally continuous tangents. The main problem of the present paper is the following: Let S be a bounded region, C the boundary of S, D_S the class of all functions which are harmonic in S, continuous in $S+C$ and possess bounded first derivatives in S and H_S^p the class of harmonic functions in S for which

$$\|h(z)\|_p\equiv\int\int_{(S)}|h(z)|^p dxdy$$

exists ($p\geqq 1$). Then it is desired to determine a class of regions, which is as wide as possible, such that the set D_S is closed in S with respect to H_S^p, that is, for every $h(z)\epsilon H_S^p$ and every $\epsilon>0$ there exists an $f(z)\epsilon D_S$ such that $\|h(z)-f(z)\|_p<\epsilon$. This problem may also be extended to unbounded regions S, in which case the definition of the class H_S^p must be somewhat modified. Any region S in which D_S is closed with respect to H_S^p is said to belong to the class (II_p). Certain special classes of such regions have been given by Zaremba [Bull. Int. Acad. Polon. Sci. (1908); Ann. École Norm. (1909)]. O. J. Farrel has solved an analogous problem for analytic functions [Bull. Amer. Math. Soc. **40**, 908–914 (1934)]. The principal results of the paper are: (1) Any star-shaped region (here used in the sense of a region bounded by a star-shaped curve) belongs to (II_p). (2) Two regions S and S' are said to be of the same conformal type if there exists a conformal map of S onto S' such that the derivative of the mapping function remains between two positive constants within a neighborhood of the boundary V of S (V contained in S; if S and S' are bounded one may take $V=S$). Then, if $S\epsilon(\mathrm{II}_p)$, any region S' of the same conformal type is also in (II_p). (3) Any region S which is

of the conformal type of a star-shaped region belongs to (II_p) (from (1) and (2)). (4) The main result of the paper: A bounded region S of finite connectivity belongs to (II_p) if its boundary C consists of a finite number of continua C_k any two of which are disjoint and each one of which determines a simply connected (bounded or infinite) region S_k of the conformal type of a star-shaped region ($S=S_0S_1\cdots S_n$). The author arrives at this result by using, in connection with (3), the following (unique) decomposition of a function $h(z)$ which is harmonic in a region S as described in (4) [cf. also Walsh, Bull. Amer. Math. Soc. **35** (1929)]:

$$h(z)=h_0(z)+\sum_{k=1}^{n}[h_k(z)+\mu_k\log|z-a_k|],$$

where a_k is a point on C_k, $h_k(z)$ is harmonic in S_k, $k=0, 1, \cdots, n$, and (assuming S_0 is the bounded region) $h_n(\infty)=0$, $k=1, 2, \cdots, n$, $2\pi\mu_k$ is the period of the conjugate of $h(z)$ along a contour which separates C_k from the other C_l. In addition, the paper contains several useful lemmas and extensions of the main result (4) to unbounded regions.

S. E. Warschawski (St. Louis, Mo.).

Keldych, M. Sur l'approximation des fonctions analytiques dans des domaines fermés. Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 137–148 (1940). (French. Russian summary) [MF 3195]

Let D denote a limited region in the z-plane. In order that every function which is regular in D and continuous in its closure $\bar{D}$ be representable in $\bar{D}$ by a uniformly convergent series of polynomials in z, it is necessary, as the author points out, that the boundary Γ of D be a closed curve in the sense of Schoenfliess. This means that Γ be the boundary of D and of exactly one other region D_∞ containing $z=\infty$. Let A and B be two points of Γ, accessible in D, and $\widehat{AB}$ a Jordan arc in D. The arc $\widehat{AB}$ decomposes D into two sub-regions D_1 and D_2. Let Γ_1 and Γ_2 denote the parts of the boundaries of D_1 and D_2 which are contained in Γ. Then A and B are said to decompose Γ into two arcs Γ_1 and Γ_2. The main result proved in the paper is the following: Let D be a region bounded by a closed curve in the sense of Schoenfliess. If there exists a decomposition of Γ into two proper sub-arcs (that is, two sub-arcs neither of which is identical with Γ), then any function which is regular in D and continuous in $\bar{D}$ can be represented as a uniformly convergent series of polynomials in $\bar{D}$. This result is an extension of the well-known theorem of Walsh [Math. Ann. **96** (1927)] on the corresponding representation in a region bounded by a closed Jordan curve. Regarding the hypothesis on Γ the author mentions an example by Brouwer [Math. Ann. **68**, 423 (1910)] of a closed curve in the sense of Schoenfliess which cannot be decomposed into two proper arcs.

S. E. Warschawski (St. Louis, Mo.).

Golusin, G. M. On complete systems of functions in the complex domain. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **6**, 48–51 (1939). (Russian) [MF 3288]

The author generalizes a classical expansion theorem of Birkhoff [C. R. Acad. Sci. Paris **164**, 942–945 (1917)] as follows. Let B be a simply connected region, not containing $z=\infty$, with more than one boundary point, let $w=\psi(z)$ map B on the interior of the unit circle, and let C_r denote the curve corresponding to $|w|=r$, $0<r<1$. If $0<r_n<1$, $r_n\uparrow 1$, $\epsilon_n>0$, and $\sum\epsilon_n<1$, and if the functions $v_n(z)$, analytic in B, satisfy

$$|v_n(z)-\psi(z)^n|\leqq r_0^n\epsilon_n$$

in C_{r_n} ($n=1, 2, \cdots$), then any function $f(z)$, analytic in B, has a unique development in a series of the functions $v_n(z)$, uniformly convergent on every closed subset of B. The proof is a generalization of Birkhoff's proof of the original theorem (in which B is a circle), using the theory of integral equations. The statement of the theorem becomes considerably simpler when B is a bounded Jordan region, and $v_n(z)$ and $f(z)$ are continuous in $\bar{B}$. [In this case a similar theorem has been given by Sheffer, Amer. J. Math. **57**, 587–614 (1935); cf. also the work of Walsh, Trans. Amer. Math. Soc. **26**, 155–170 (1924); **31**, 53–57 (1929).] The author points out that a result of Heuser [Math. Ann. **110**, 1–11 (1934); he misprints the name as "Heisel"] on polynomial expansions follows as a corollary.

R. P. Boas, Jr. (Durham, N. C.).

Schaginjan, A. Sur l'approximation au moyen de polynômes dans les domaines non-jordaniens. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 318–320 (1940). [MF 3245]

The note contains two remarks the first of which is rather obvious. (1) Let γ be a Jordan arc, c a cut from a point of γ into its interior, and let $C=\gamma+c$ be rectifiable. Let $f(z)$ be analytic in the interior of C and

$$\int_{C_r}|f(z)|^2d\sigma=O(1) \qquad \text{as } r\to 1-0;$$

here C_r is the level curve of an interior mapping of C onto the unit circle. Then, as is well known, $f(z)$ possesses almost everywhere on C boundary values for which $|f(z)|^2$ is integrable on C. The author shows that, if $f(z)$ can be approximated by polynomials $p_n(z)$ in the sense

$$\int_C|f(z)-p_n(z)|^2d\sigma\to 0,$$

$f(z)$ must be regular in the interior of γ. (2) New proof of a result of Keldych stating that in a simply connected domain B analytic functions $f(z)$ exist for which

$$\iint_B|f(z)|^2dxdy, \qquad z=x+iy,$$

is finite and yet the approximation

$$\iint_B|f(z)-p_n(z)|^2dxdy\to 0$$

by polynomials $p_n(z)$ is impossible.

G. Szegö.

Montel, Paul. Sur quelques propriétés des couples de fonctions uniformisantes. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 166–171. [MF 3511]

If $F(x, y)$ is a polynomial of the complex variables x, y and if $f(z)$, $g(z)$ are meromorphic functions of z in some circle $|z|<R$ and regular at $z=0$, then the pair $f(z)$, $g(z)$ is said to furnish a local uniformization of the algebraic curve $F(x, y)=0$ around its point (a_0, b_0) if $f(0)=a_0$, $g(0)=b_0$ and $F(f(z), g(z))\equiv 0$ for $|z|<R$. According to Picard, if the genus of the curve exceeds unity, then the radius R is finite. The purpose of the paper is to show that this theorem, as well as various generalizations, can be obtained in a simple and natural way by using the theory of normal families originated by the author.

T. Radó (Columbus, Ohio).

Cotlar, Mischa. Normal families of non-analytic functions. An. Soc. Ci. Argentina **129**, 3–25 (1940). (Spanish) [MF 3772]

The function $w=f(z)=u(x, y)+iv(x, y)$ is said to be polygenic if the derivatives u_x, u_y, v_x, v_y exist and are continuous in the domain of definition D. Such a function admits a radial derivative in every direction at each point of D, as one easily verifies:

$$\left(\frac{dw}{dz}\right)_\rho = Sf(z)+e^{-2i\rho}Af(z).$$

Here $Sf(z)$ and $Af(z)$ are called the symmetric and the areal derivatives of $f(z)$, respectively. A polygenic function is holomorphic if and only if its areal derivative vanishes identically. By means of inequalities between higher symmetric and areal derivatives, the author defines a non-vacuous class (NG) of functions more general than the class of holomorphic functions but less general than the class of polygenic functions. Many strong theorems of analytic function theory generalize to functions of class (NG): theorems concerning the order of a zero, the number of zeros in a region, the maximum modulus principle, the theorems of Bloch, Schotky, Picard, etc. *E. F. Beckenbach.*

Bergman, Stefan. The visualization of domains of the theory of functions of two complex variables. J. Math. Phys. Mass. Inst. Tech. **20**, 107–117 (1941). [MF 3846]

The paper is an expository paper in which a method of visualizing four-dimensional space is presented. The method consists in interpreting the fourth dimension as the time coordinate and thus visualizing a given four-dimensional object by means of three-dimensional moving figures. The method is clearly illustrated by the judicious choice of several examples, both in four-dimensional space and in lower dimensional space. In the second part of the paper the author explains the notion of a (four-dimensional) domain with a "distinguished boundary surface" by the above method and proves one of the function-theoretic properties of these domains. *W. T. Martin* (Princeton, N. J.).

Tornehave, Hans. On the theory of domains of regularity. Mat. Tidsskr. B. **1940**, 45–61 (1940). (Danish) [MF 3880]

This is a lecture giving a review of recent advances in the theory of domains of regularity of analytic functions of two or more complex variables, due mainly to H. Cartan, K. Oka, K. Stein, P. Thullen and A. Weil. *B. Jessen.*

Fourier Series and Integrals, Theory of Approximation

***Churchill, Ruel V. Fourier Series and Boundary Value Problems.** McGraw-Hill Book Company, Inc., New York and London, 1941. ix+206 pp. $2.50.

List of contents: Ch. I. Introduction. Ch. II. Partial differential equations of physics. Ch. III. Orthogonal sets of functions. Ch. IV. Fourier series. Ch. V. Further properties of Fourier series. Fourier integrals. Ch. VI. Solution of boundary value problems by the use of Fourier series and integrals. Ch. VII. Uniqueness of solution. Ch. VIII. Bessel functions and applications. Ch. IX. Legendre polynomials and applications.

Fedoroff, V. S. Sur les coefficients de Fourier. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 127–128 (1940). [MF 3545]

A summary of results published in detail in Rec. Math. [Mat. Sbornik] N.S. **8 (50)**, 41–56 (1940); cf. these Rev. **2**, 94. *A. Zygmund* (South Hadley, Mass.).

Kharchiladze, F. J. Quelques critères nouveaux pour la détermination de la classe d'une série trigonométrique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 137–138 (1940). [MF 3548]

Necessary and sufficient conditions for a trigonometric series to be the Fourier series of a function of various classes have been obtained in terms of Cesàro and Abel means formed from the series [see Zygmund, Trigonometrical Series, Warsaw-Lwów, 1935, pp. 78–86]. The author states corresponding criteria in terms of the Bernstein means [S. Bernstein, C. R. Acad. Sci. Paris **191**, 976–979 (1930)]

$$B_n(x)=\tfrac{1}{2}\{S_n(x)+S_n[x+2\pi/(2n+1)]\},$$

where $S_n(x)$ is the nth partial sum of the series in question. These criteria, for the classes M (bounded functions), C (continuous functions), L^p and V (Fourier-Stieltjes series of functions of bounded variation), have the same form as in the Cesàro case, with the arithmetic means of the $S_n(x)$ replaced by the $B_n(x)$. The author states that they may be established by reasoning analogous to that used in the Cesàro case, using theorems (which he quotes) previously established by Bernstein and by Natanson [see, in particular, Bernstein, l.c., and Natanson, C. R. (Doklady) Acad. Sci. URSS (N.S.) **19**, 357–360 (1938)]. *R. P. Boas, Jr.*

Menchoff, D. Sur les séries de Fourier des fonctions continues. Rec. Math. [Mat. Sbornik] N.S. **8 (50)**, 493–518 (1940). (French. Russian summary) [MF 3746]

The problem whether the Fourier series of continuous functions converges almost everywhere remains unsolved. In the present paper the author shows that to every continuous function $f(x)$ and to every $\sigma>0$ there exists a continuous function $g(x)$ such that (a) $f(x)=g(x)$ in the interval $(0, 2\pi)$, except for a set of measure less than σ; (b) the Fourier series of $g(x)$ converges almost everywhere. In view of the very well-known Lusin theorem on measurable functions, the above result holds if $f(x)$ is any measurable function, finite almost everywhere in $(0, 2\pi)$.

A. Zygmund (South Hadley, Mass.).

Menchoff, D. Sur la représentation des fonctions mesurables par des séries trigonométriques. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 214–216 (1940). [MF 3552]

Lusin proved [C. R. Acad. Sci. Paris **154**, 1688–1690 (1912)] that for every measurable function $f(x)$ there exists a continuous function $F(x)$ such that $F'(x)=f(x)$ almost everywhere. If $F(x)$ is defined in an interval of length 2π, then, as Lusin himself observed, the Fourier series of $F(x)$ differentiated term by term gives a trigonometric series summable $(C, 2)$ to the function $f(x)$ for almost every x. Menchoff now generalizes this result by showing that for every measurable function $f(x)$ of period 2π there exists a trigonometric series convergent to $f(x)$ almost everywhere. The representation is of course not unique.

A. Zygmund (South Hadley, Mass.).

Randels, W. C. On the absolute summability of Fourier series, III. Duke Math. J. **7**, 204–207 (1940). [MF 3398]

As an analogue to a theorem of N. Wiener on absolute

convergence, the following theorem is proved: If $f(x)$ is such that to any point y on the closed interval $(-\pi, \pi)$ there exist a function $g_y(x)$ and a $\delta>0$ such that $g_y(x)=f(x)$ for $|x-y|<\delta$ and the Fourier series of $g_y(x)$ is absolutely summable $|C, 1|$, then the Fourier series of $f(x)$ is absolutely summable $|C, 1|$ on $(-\pi, \pi)$. *O. Szász.*

Raikov, D. Sur les fonctions positivement définies. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 860–865 (1940). [MF 3580]

Very elegant treatment of positive definite functions on the line $-\infty<t<\infty$, including a new proof of the theorem of Krein that any function which is positive definite on an interval $-\omega<t<\omega$ can be continued (not necessarily uniquely) onto the whole line. *S. Bochner.*

Szász, Otto. The jump of almost periodic functions and of Fourier integrals. Duke Math. J. **7**, 360–366 (1940). [MF 3407]

Let $\psi(t)$ be a B.a.p. function satisfying the condition $\psi(-t)=-\psi(t)$, and let

$$\psi(t)\sim\sum b_\nu \sin \lambda_\nu t, \qquad \lambda_\nu>0.$$

If there exists a number D such that

$$(*) \qquad \int_0^h |\psi(t)-D|\,dt=o(h) \qquad \text{as } h\to 0,$$

then the expression

$$\frac{1}{\eta}\sum_{0<\lambda_n\le\eta}\lambda_n b_n+\sum_{\eta<\lambda_n<2\eta}\left(2-\frac{\lambda_n}{\eta}\right)b_n$$

tends to $(4/\pi)D\log 2$ as $\eta\to\infty$. This result permits the determination of the jumps of a B.a.p. function by means of its coefficients, and is a generalization of a similar result for Fourier series [see O. Szász, Duke Math. J. **4**, 401–407 (1938)]. Furthermore the condition (*) on D is relaxed and the result is extended to Fourier's integral.

A. Zygmund (South Hadley, Mass.).

Kovanko, A. S. Sur les systèmes compacts de fonctions presque-périodiques généralisées de W. Stepanoff. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 211–213 (1940). [MF 3551]

The author shows that Stepanoff's almost periodic functions and "Stepanoff normal functions" are equivalent. He thus generalizes the theorem that Bohr's almost periodic functions and Bochner's normal functions are equivalent. The paper also deals with the compactness of systems of S.a.p. functions. *R. H. Cameron* (Cambridge, Mass.).

Lewitan, B. Verallgemeinerte Operation der Verschiebung im Zusammenhang mit fastperiodischen Funktionen. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 631–634 (1940). [MF 3572]

This is a more condensed presentation of a paper published in the Rec. Math. [Mat. Sbornik] N.S. **7** (**49**), 449–478 (1940) [these Rev. **2**, 96]. It appears to be a German translation of the Russian summary appended to that paper. *R. H. Cameron* (Cambridge, Mass.).

Taylor, P. R. The functional equation for Epstein's zeta-function. Quart. J. Math., Oxford Ser. **11**, 177–182 (1940). [MF 3127]

A proof of the functional equation for Epstein's zeta-function

$$f(s, A, B, C)=\sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(Am^2+Bmn+Cn^2)^s},$$

is given in the special case where the quadratic form is positive definite. This proof depends on the functional equation for Riemann's zeta-function and some properties of hypergeometric functions, and it uses Mellin's inversion formula. A result, due to Kober, that expresses $f(s, A, B, C)$ as a double series involving Bessel functions is also given under the same restriction. The author implies that his proof can be extended beyond the special case considered. *H. S. Zuckerman* (Seattle, Wash.).

Herriot, John G. Cesàro summability of ordinary double Dirichlet series. Bull. Amer. Math. Soc. **46**, 920–929 (1940). [MF 3449]

The purpose of the paper is to extend to double Dirichlet series the well-known results Bohr obtained [Nachr. Ges. Wiss. Göttingen **1909**, 247–262] for simple series. The author studies the summability (C, r, s), where r, s are non-negative integers, establishes the existence and certain properties of the associate abscissas of summability, and proves the analyticity of the functions represented by summable Dirichlet series. [See also the paper of Obrechkoff reviewed below.] *A. Zygmund* (South Hadley, Mass.).

Obrechkoff, N. Sur la sommation des séries multiples de Dirichlet et des séries semblables. Ann. Univ. Sofia. Fac. Sci. Livre 1, Math. Phys. **36**, 1–145 (1940). (Bulgarian. French summary) [MF 3358]

The purpose of the paper is to extend to double series the theory of M. Riesz's typical means. The double series $\sum\sum a_{mn}$ is said to be summable $(R, \lambda, \mu; k, p)$ to sum s (where $k\ge 0$, $p\ge 0$, $\lambda_n\uparrow\infty$, $\mu_n\uparrow\infty$) if the expression

$$(*) \qquad \varphi(x, y)=x^{-k}y^{-l}\sum_{\lambda_m<x}\sum_{\mu_n<y}a_{mn}(x-\lambda_m)^k(y-\mu_n)^l$$

is bounded for all positive x, y, and tends to s as x, y tend to $+\infty$. Among others the following theorems are established: (i) If a double series is summable $(R, \lambda, \mu; k, p)$ to s, then it is also summable $(R, \lambda, \mu; k_1, p_1)$, where $k\ge k_1$, $p_1\ge p$, to s. (ii) If a double series is summable $(R, \lambda, \mu; k, p)$, then it is also summable $(R, l, m; k, p)$, where $l_n=e^{\lambda_n}$, $m_n=e^{\mu_n}$, to the same sum. (iii) Suppose that a series $\sum\sum a_{mn}$ is absolutely summable $(R, \lambda, \mu; k, p)$, which means that the integral (cf. (*))

$$\int_0^\infty\int_0^\infty\left|\frac{\partial^2\varphi(x, y)}{\partial x\partial y}\right|dxdy$$

is finite. Then the series is absolutely summable $(R, \lambda, \mu; k_1, p_1)$ for $k_1\ge k$, $p\ge p_1$. (iv) If the double Dirichlet series

$$\sum\sum a_{mn}e^{-\lambda_m s-\mu_n u}$$

is summable $(R, \lambda, \mu; k, p)$ at some point s_0, u_0, then it is summable by the same method in the half planes $\Re s>\Re s_0$, $\Re u>\Re u_0$ (this holds also for absolute summability) and represents there a regular function.

Moreover, the author proves certain theorems for the generalized typical means defined by the expressions

$$\frac{1}{\psi(x)\chi(y)}\sum_{\lambda_m<x}\sum_{\mu_n<y}a_{mn}\psi(x-\lambda_m)\chi(y-\mu_n),$$

analogous to (*) (where ψ, χ satisfy certain conditions of regularity), studies the behavior of Newton double series,

and establishes a number of theorems on the Euler-Knopp method of summation of double series. Certain aspects of the theory of double Dirichlet series have also been investigated in a recent paper of Herriot [see the preceding review]. *A. Zygmund* (South Hadley, Mass.).

Obrechkoff, Nikola. Sur les développements asymptotiques et la transformation de Laplace. Ann. Univ. Sofia. Fac. Sci. Livre 1, Math. Phys. **36**, 171–200 (1940). (Bulgarian. French summary) [MF 3356]

One of the results obtained by the author is as follows. Let G be a closed region bounded by a simple contour C and containing the sector $s-\alpha=\rho e^{i\phi}$, $-\delta\leqq\phi\leqq\delta_1$, $\pi/2<\delta<\pi$, $\pi/2<\delta_1<\pi$. Let $f(s)$ be analytic in G and let $f(s)\to 0$ uniformly when s tends to infinity, remaining in G. Assume also that the integral

$$\int_C \frac{f(\zeta)d\zeta}{\zeta-a}$$

exists, when a is a suitable complex number. Then $f(s)$ admits of a representation

$$f(s)=\int_0^\infty e^{-sx}F(x)dx,$$

where

$$F(x)=(2\pi i)^{-1}\int_C e^{sx}f(s)ds=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} e^{sx}f(s)ds,$$

where c is any real number greater than σ_0, the least upper bound of real parts of singularities of $f(s)$. The function $F(x)$ is analytic in the sector G': $\pi/2-\delta_1<\arg x<\delta-\pi/2$ and admits of an asymptotic expansion

$$F(x)\sim\frac{a_0}{\Gamma(\lambda_0+1)}x^{\lambda_0}+\frac{a_1}{\Gamma(\lambda_1+1)}x^{\lambda_1}+\cdots \quad \text{as } |x|\to 0,$$

whenever $f(s)$ admits of an asymptotic expansion

$$f(s)\sim\frac{a_0}{s^{\lambda_0+1}}+\frac{a_1}{s^{\lambda_1+1}}+\cdots \qquad \text{as } |s|\to\infty.$$

An analogous theorem is proved for the case where $f(s)$ admits of an asymptotic expansion as $|s|\to 0$. These results are analogous to those obtained by Mellin [Ann. Acad. Sci. Fennicae (A) **18**, no. 4 (1922)]. The author also derives an expansion of the form $F(z)=\sum_{-\infty}^{\infty}c_n e^{n\zeta}$ for a function $F(z)$ which is meromorphic and periodic (of period 2π), having only simple poles and regular on the axis of reals. It should be observed that a theorem proved by the author concerning the differentiability of the asymptotic expansion of an analytic function is well known [see, e.g., R. Nevanlinna, Ann. Acad. Sci. Fennicae (A) **12**, no. 3 (1919)]. *J. D. Tamarkin* (Providence, R. I.).

Avakumović, Vojislav G. Über das Verhalten Dirichletscher Reihen am Rande des Konvergenzgebietes. Math. Z. **46**, 650–664 (1940). [MF 3374]

Proof of function theoretical Tauberian theorems. The following is typical. Let

$$J(s)=s\int_0^\infty e^{-su}A(u)du$$

converge for $\Re(s)>0$. Let $J(s)$ be bounded in a convex domain D_k in $\Re(s)>0$ which has contact of order $k-1>0$ with the imaginary axis at $s=0$. Let

$$\lim_{y\to\infty}\lim_{\delta\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin yt}{t}J(\delta+|t|^k+iat)dt=A, \tag{1}$$

$a\neq 0$ real. Finally a (C_k-o) condition:

$$\liminf_{u\to\infty}\min_{u\leq u_1\leq\omega}[A(u_1)-A(u)]=-w(\epsilon)\to 0 \tag{2}$$

as $\epsilon\to 0$, where $\omega=u+\epsilon u^{1/k}$. Then $A(u)\to A$ as $u\to\infty$. If in (1) the limit with respect to y be replaced by lim inf and lim sup, the results being $O(1)$, and if (2) be replaced by a (C_k-O) condition: $A(u_1)-A(u)>-w(\epsilon)$ for some $\epsilon>0$, then the conclusion becomes $A(u)=O(1)$ as $u\to\infty$ instead. For the history of the paper, preliminary communications, and the relation to the work of N. Wiener and H. R. Pitt, see the footnote on p. 651. *E. Hille.*

Agnew, Ralph Palmer. Tauberian conditions. Ann. of Math. (2) **42**, 293–308 (1941). [MF 3685]

The author considers Tauberian conditions as such. He shows that the several classes he considers are contained in a class of H. R. Pitt for which Pitt proved the Fourier transform method of Wiener applicable. The author then considers a more general condition to which he states the methods of Wiener and Pitt probably are applicable. *N. Levinson* (Cambridge, Mass.).

Kendall, M. G. Conditions for uniqueness in the problem of moments. Ann. Math. Statistics **11**, 402–409 (1940). [MF 3419]

The uniqueness question for the problem of moments in $(-\infty, \infty)$ is considered along the lines of the criteria of Carleman and P. Lévy. "The purpose of this paper is to develop criteria of this kind more systematically and to give more general criteria. . . ." In fact the author obtains Carleman's theorem and some very simple special cases of it. The only new lemma [p. 405 above] is wrong. The proofs are surprisingly short: unfortunately they depend essentially on the assertion that $\sum a_n$ always converges if $a_n\downarrow 0$ and $na_n\to 0$. There are more errors of this kind. *W. Feller* (Providence, R. I.).

Bödewadt, U. T. Über Funktionen mit n Zeichenwechseln, deren Momente bis zur Ordnung $n-1$ verschwinden. Jber. Deutsch. Math. Verein. **50**, 129–139 (1940).

Let $f(x)$ be of bounded variation in $[0, 1]$. Let $\mu_n=\int_0^1 f(x)x^n dx$, $n=0, 1, 2, \cdots$. The author calls $f(x)$ extremal (Randfunktion) of order n if $\mu_0=\mu_1=\cdots=\mu_{n-1}=0$, while $f(x)$ has precisely n changes of sign. It is proved that the moments of the nth and higher order of an extremal function of order n are different from zero and have the sign of $f(x)$ in the neighborhood of $x=1$. The author further proves various propositions concerning the decomposition of an extremal function of order n into sums of extremal functions of smaller orders, and the resulting expressions for the moments of order higher than $(n-1)$. *J. D. Tamarkin.*

Kober, H. On fractional integrals and derivatives. Quart. J. Math., Oxford Ser. **11**, 193–211 (1940). [MF 3129]

The author introduces four operators of generalized fractional integration

$$L^*_{\eta,\alpha}[f;x]=\frac{1}{\Gamma(\alpha)}x^\beta\int|t-x|^{\alpha-1}t^\gamma f(t)dt.$$

In the author's notation L is either I or K, $*$ is either $+$

or $-$. For $L=I$, $\beta=-\alpha-\eta$, $\gamma=\eta$; for $L=K$ these values should be interchanged. The range of integration is $(0, z)$ or (z, ∞) according as * is $+$ or $-$. The formal relations between these operators and the fractional integrals of Weyl and Hardy-Littlewood are studied, as well as their composition and integrability properties. It is usually assumed that $f(t)\varepsilon L_p(0, \infty)$ for some $p\geqq 1$. This implies, for instance, that $z^{1/p-1/q}I^+_{\eta,\alpha}[f; z]$ is a bounded transformation on L_p to L_q if $\Re(\zeta)>-1/p'$ and if either $1\leqq p\leqq\infty$, $q=p$, or $1<p<q<\infty$, $1/p-1/q\leqq\Re(\alpha)\leqq 1/p$, or $1\leqq p\leqq q\leqq\infty$, $\Re(\alpha)>1/p$. Formulas for integration by parts of the type

$$\int_0^\infty I^+_{\eta,\alpha}[f; z]g(z)dz=\int_0^\infty K^-_{\eta,\alpha}[g; z]f(z)dz$$

are proved under suitable restrictions. The Mellin transforms of $I^+_{\eta,\alpha}[f]$ and $K^-_{\eta,\alpha}[f]$ are expressed as gamma quotients times the Mellin transform of $f(t)$ when $f(t)\varepsilon L_p(0, \infty)$ and $1\leqq p\leqq 2$. The formulas are also valid for $2<p$ provided $f(t)$ is the inverse Mellin transform of a function in $L_{p'}$ $(-\infty, \infty)$. This result extends to Mellin transforms of more general transforms than $I^+_{\eta,\alpha}[f]$ involving a kernel $K(x, y)$ which is homogeneous and of degree -1. The author gives new proofs for certain theorems concerning fractional derivatives (uniqueness, existence of derivatives of lower order, etc.). A necessary and sufficient condition that the equation $g(z)=I^+_{\eta,\alpha}[f; z]$, $\Re(\eta)>-1/p'$, shall have a solution $f(t)\varepsilon L_2(0, \infty)$ is that $g(z)$ and $t^\alpha g_1(t)\varepsilon L_2(0, \infty)$, where $g_1(t)$ is the Mellin transform of $g(z)$. Corresponding conditions become merely necessary if $1\leqq p<2$ but remain sufficient if $2<p\leqq\infty$. Similar results hold for $K^-_{\eta,\alpha}[f]$. The case $\eta=0$ in $I^+_{\eta,\alpha}$, $\eta=-\alpha$ in $K^-_{\eta,\alpha}$, give results concerning ordinary fractional differentiation. Finally the operators are expressed as products of two operators of the Hankel type [see also the following review]. *E. Hille* (New Haven, Conn.).

Erdélyi, A. and Kober, H. Some remarks on Hankel transforms. Quart. J. Math., Oxford Ser. 11, 212–221 (1940). [MF 3130]

Hankel transforms are here defined as

$$\mathfrak{H}_\nu[f]=\int_0^\infty J_{\nu,m}[2(tx)^{\frac{1}{2}}]f(t)dt, \qquad f(t)\varepsilon L_p(0, \infty);\ 1\leqq p\leqq 2,$$

where the integral is the limit in the mean of index p' when the upper limit tends to ∞, and

$$J_{\nu,m}(z)=\frac{(-1)^m(\frac{1}{2}z)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)}\,{}_1F_2(1; m+1, \nu+m+1; -\tfrac{1}{4}z^2).$$

Further $m=0$ when $\frac{1}{2}\Re(\nu)>-1/p'$, otherwise m is the positive integer for which $1/p-1<\frac{1}{2}\Re(\nu)+m<1/p$. It is shown that a Parseval theorem and a Hausdorff-Young theorem hold for $\mathfrak{H}_\nu[f]$. The main part of the paper is devoted to a discussion [for the notation see the preceding review] of relations of the form $\mathfrak{H}_{\nu+2\alpha}L^*_{\frac{1}{2}\nu,\alpha}=L^*_{\frac{1}{2}\nu,\alpha}\mathfrak{H}_\nu$ and corresponding relations for the commuted operator products. The results are rather complete when $p=2$, less so for $1\leqq p<2$. For the detailed conditions, which are rather complicated, we must refer to the paper itself. *E. Hille.*

Erdélyi, A. On fractional integration and its application to the theory of Hankel transforms. Quart. J. Math., Oxford Ser. 11, 293–303 (1940). [MF 3792]

Kober has introduced four operators of generalized fractional integration, $I^\pm_{\eta,\alpha}$ and $K^\pm_{\eta,\alpha}$, and Kober and Erdélyi have applied them to the theory of Hankel transforms [see the preceding two reviews]. These operators applied to functions of $L_p(0, \infty)$ only when the complex parameter η was restricted to a half plane (depending on p). Erdélyi now extends the operators I and K by modifying their kernels so that they will be defined for all complex values of η except those on an infinite set of lines $\Re(\eta)=$constant. The idea of the modification is that, for example, when $\Re(\eta)<-1/p'$ and $f(u)\varepsilon L_p(0, \infty)$, we are unable to integrate $u^\eta f(u)$ on $(0, x)$; so we first integrate on (x, ∞) h times, where h is a suitable positive integer, and then integrate $(\alpha-h)$ times on $(0, x)$, thus defining an integral of order α. The definition which results is

$$I^+_{\eta,\alpha}[f; z]$$
$$=\frac{z^{-\eta-\alpha}}{\Gamma(\alpha)}\left\{\int_0^z\left[(z-u)^{\alpha-1}-\sum_{r=0}^{h-1}\binom{\alpha-1}{r}z^{\alpha-r-1}(-u)^r\right]u^\eta f(u)du\right.$$
$$\left.-\int_z^\infty\left[\sum_{r=0}^{h-1}\binom{\alpha-1}{r}z^{\alpha-r-1}(-u)^r\right]u^\eta f(u)du\right\},$$

where $\Re(\alpha)>0$, $h=0$ if $\Re(\eta)>-1/p'$, and otherwise h satisfies $-1/p'<\Re(\eta)+h<1/p$. The operator K is defined similarly (with the roles of the finite and infinite intervals interchanged). The results generalize those of Kober mentioned in the first of the two preceding reviews. In the second part of the paper, the operators $I_{\eta,\alpha}$ and $K_{\eta,\alpha}$, as applying to functions of L_2, are extended to all complex values of α, and the results of Erdélyi and Kober on Hankel transforms [see the preceding review] are generalized in the case $p=2$. All the results of Erdélyi and Kober connecting Hankel transforms of different orders can now be included in one relatively simple theorem (which contains other results as well). *R. P. Boas, Jr.* (Durham, N. C.).

Erdélyi, A. A class of hypergeometric transforms. J. London Math. Soc. 15, 209–212 (1940). [MF 3140]

Certain classes of transformations T_α introduced by C. Fox [J. London Math. Soc. 14, 278–281 (1939); these Rev. 1, 51] are shown to be expressible in terms of the fractional integrals $I^+_{\eta,\alpha}[f]$ and $K^-_{\eta,\alpha}[f]$ of Kober and in terms of Hankel transforms. Let $R[f]=(1/x)f(1/x)$ and

$$T_\alpha=(I^+_{\frac{1}{2}\nu,\alpha})^{-1}RI^+_{\frac{1}{2}\nu,\alpha}=I^+_{\frac{1}{2}\nu+\alpha,-\alpha}K^-_{\frac{1}{2}\nu,\alpha}R$$

for $\Re(\alpha)\geqq 0$, $\Re(\nu)$ and $\Re(\nu+2\alpha)$ not negative odd integers. If $f(x)\varepsilon L_2(0, \infty)$ so does $F(x)=T_\alpha[f]$, $T_\alpha^{-1}=T_{-\alpha}$, $T_0=R$, and $\lim_{\alpha\to 0}(1/\alpha)F(x/\alpha^2)=\mathfrak{H}_\nu[f]$. Similar definitions are given for $\Re(\alpha)<0$. In particular, for $\Re(\nu)>-1$,

$$T_n[f]=(-1)^nx^{-\frac{1}{2}}f(x^{-1})+\int_0^{1/x}k(xu)f(u)du,$$

$$k(x)=\frac{\Gamma(\nu+n+1)}{\Gamma(n)\Gamma(\nu+1)}x^{\frac{1}{2}\nu}\,{}_2F_1(1-n, \nu+n-1; \nu+1; x).$$

Finally $T_\alpha=\mathfrak{H}_\nu\mathfrak{H}_{\nu+2\alpha}R$. *E. Hille* (New Haven, Conn.).

Dhar, S. C. On certain self-reciprocal functions. J. Indian Math. Soc. (N.S.) 4, 91–96 (1940). [MF 3271]

The author shows that the functions

$$x^{\nu-2k+\frac{1}{2}}e^{-\frac{1}{2}x^2}W_{k,\pm(k-\frac{1}{2})}(\tfrac{1}{2}x^2),$$
$$x^{\nu\mp 2m-\frac{1}{2}}e^{\frac{1}{2}x^2}W_{-\nu\pm 2m-k,m}(\tfrac{1}{2}x^2)$$

are R_ν, that is, self-reciprocal in the Hankel transform of order ν. Neither of these is new, since the first reduces to the simple R_ν function

$$x^{\nu+\frac{1}{2}}e^{-\frac{1}{2}x^2},$$

and the second has been published by W. N. Bailey [J. London Math. Soc. 5, 263 (1930)]. *M. C. Gray.*

Graves, Lawrence M. Some general approximation theorems. Ann. of Math. (2) **42**, 281–292 (1941). [MF 3684]

This paper is concerned with some general approximation theorems which include as special cases well-known theorems on the approximation of a given function by polynomials, by trigonometric sums, and by integral means. The proofs are so formulated that there is little more complication in the case of functions of m variables than in the case of functions of one variable. The author gives theorems on the term-by-term integration and differentiation of the approximating functions, and also considers certain cases of iteration of the approximating transformations. For a given $f(x)$ which is continuous on a bounded closed set A_0, and which vanishes on a subset A_1 of A_0, it is established that there exist certain types of approximating functions which vanish on a neighborhood of A_1 and are of class $C^{(r)}$, where r is a preassigned positive integer. *W. T. Reid.*

Bernstein, S. Sur la meilleure approximation locale des fonctions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 843–846 (1940). [MF 3576]

In order that a function $f(x)$ have a continuous derivative of order $n+1$ on an interval (a, b) it is shown to be necessary and sufficient that, if x_0 is an arbitrary point of (a, b), if (α, β) is an infinitesimal subinterval whose ends approach x_0, and if $E_n[f(x); (\alpha, \beta)]$ is the minimum error of approximation to $f(x)$ on (α, β) by a polynomial of degree n, the ratio $E_n/|\beta-\alpha|^{n+1}$ as a function of α and β approaches uniformly a limit which is a constant multiple of $|f^{(n+1)}(x_0)|$; the value of n is fixed throughout. The necessity of the condition is recognized almost immediately. The proof of sufficiency depends on a lemma of D. Raikov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 653–656 (1939); these Rev. **1**, 333]. *D. Jackson* (Minneapolis, Minn.).

Zoukhovitzky, S. Sur une fonction qui s'écarte asymptotiquement le moins possible de zéro sur un intervalle fixe. Acad. Sci. RSS Ukraine. Rec. Trav. [Zbirnik Prace] Inst. Math. **1940**, no. 4, 175–179 (1940). (Ukrainian. Russian and French summaries) [MF 3236]

A solution is given, by well-known methods, of the following problem. Among all functions of the type

$$\left[\sum_{i=0}^{k} \sigma_i x^{n+l-i} + \sum_{j=k+1}^{n+l} p_j x^{n+l-j}\right] \prod_{j=1}^{2l} (x-a_j)^{-1},$$

where l, k, σ_i, a_j are given, the latter, real or conjugate complex, not belonging to $[-1, 1]$, find asymptotically $(n \to \infty)$ the one which deviates the least from zero in $[-1, 1]$ and its deviation. *J. A. Shohat* (Philadelphia, Pa.).

McEwen, W. H. On the degree of convergence of the derived series of Birkhoff. Amer. J. Math. **63**, 29–38 (1941). [MF 3626]

It is known that the remainder after terms of the Nth order in the Birkhoff series for a function $f(x)$ is uniformly $O(1/N^m)$ if $f(x)$ has a continuous mth derivative of limited variation, and if $f(x)$ and its first $m-1$ derivatives vanish at the ends of the interval in question [W. E. Milne, Trans. Amer. Math. Soc. **19**, 143–156 (1918)]. The author has shown recently that under suitable hypotheses the successive derived series, up to a specified order of differentiation, converge to the corresponding derivatives of $f(x)$ [Bull. Amer. Math. Soc. **45**, 576–582 (1939); these Rev. **1**, 11]. On the basis of the latter result it is now shown that under the hypotheses first stated the remainder in the kth derived series is uniformly $O(1/N^{m-k})$ for $k=1, 2, \cdots, m-1$. The same degree of convergence is obtained on a closed interior interval by an appropriate method of summation, without the assumption that $f(x)$ and its derivatives vanish at the end points. *D. Jackson* (Minneapolis, Minn.).

Taldykin, A. Uniformly minimal systems of functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 531–534 (1940). [MF 3560]

If (1) $\{\phi_i\}$ is a minimal set of functions of L^2 and (2) $\{\phi^i\}$ is the adjoined set, which forms with (1) a biorthogonal system, the author calls the set (1) uniformly minimal if the norms of (1) and (2) are both uniformly bounded. He shows that (1) is uniformly minimal if and only if the lower bound of the set of characteristic roots of all the Gramian matrices $(\int \phi_i \phi_j)$, $i, j=1$ to n, is positive. *O. Frink.*

Taldykin, A. Normal systems of functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 535–539 (1940). [MF 3561]

A system (1) $\{\phi_i\}$ of functions of L^2 of bounded norm is called normal if the set of all characteristic roots of the finite Gramian matrices of the system is bounded and has a positive lower bound. A normal system is uniformly minimal [see the preceding review] and is part of a biorthogonal system $\{\phi_i, \phi^i\}$. The system adjoined to a uniformly minimal system is always normal. It is shown that the infinite Gramian matrices Φ and Φ' of a normal system and its adjoined system are both left and right inverses of each other, and that a uniformly minimal system is normal if and only if the spaces of the system and of its adjoined system coincide. Several properties of orthogonal systems are shown to hold also for normal systems. The biorthogonal series expansions of any function f of L^2 in terms of a normal system and in terms of the adjoined system both converge in the mean to the same function, which is the projection of f on the common space of the two systems. An analogue for normal systems of the Riesz-Fischer theorem is shown to be a consequence of this result. The proof of an extension of the Rademacher-Menchoff theorem to normal systems is sketched. *O. Frink* (State College, Pa.).

Taldykin, A. Integral equations with normal and semi-normal nuclei. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 540–545 (1940). [MF 3562]

The author's results on normal systems of functions [see the preceding review] are applied to integral equations of Fredholm type. A real kernel of summable square is said to be semi-normal if the left characteristic functions $\{\phi_i\}$ or the right characteristic functions $\{\psi_i\}$ form a normal system, and the kernel is called normal if both left and right systems are normal. It is shown that the kernel is normal if and only if $\phi_i = \sum \bar{\psi}_k \int \phi_i \bar{\phi}_k$ and $\psi_i = \sum \phi_k \int \psi_i \bar{\psi}_k$. The resolvent kernel of a semi-normal kernel is shown to have only simple poles.

The theorem of Hilbert and Schmidt for symmetric kernels is extended to normal kernels, and it is proved that the series expansions, in terms of both left and right characteristic functions, of a function f representable as a transform by a normal kernel converges absolutely and uniformly to the projection of f on the common space of the two systems of functions. It is also proved that a kernel N sym-

metrizable from the right by means of a definite symmetric kernel is semi-normal from the left, and, in case the definite symmetric kernel is the kernel of Schmidt, then the left characteristic functions of N are orthogonal, and the characteristic values of N are real. *O. Frink.*

Taldykin, A. On the closedness of biorthogonal systems of functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 546–549 (1940). [MF 3563]

A condition of N. Günther that a system of orthogonal functions be closed is extended to normal systems. It is shown that, if (1) $\{\phi_k\}$ is a normal system of functions of L^2, and (2) $\{\phi^k\}$ is the adjoined system, then the systems (1) and (2) are closed in L^2 if and only if for every pair of measurable sets A and B of the domain of definition $\sum\int_A\phi_k\int_B\phi^k=m(AB)$. This is the Parseval relation for the characteristic functions of the sets A and B.

O. Frink (State College, Pa.).

Rémès, E. On some estimates of best approximation and, in particular, on a fundamental theorem of de la Vallée-Poussin. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 235–244. (Russian) [MF 3515]

Let ρ denote the best approximation, in the sense of Tschebycheff, of a continuous function $f(x)$, by polynomials of degree not greater than n, on a given interval (a, b). The author uses divided differences and obtains a somewhat improved form of the lower bound for ρ as given by de la Vallée-Poussin. The possible extension of the result to approximations of a more general nature is briefly indicated. Applications are made to estimating ρ in case $f^{(n+1)}(x)$ exists on (a, b). *J. A. Shohat* (Philadelphia, Pa.).

Rémès, E. J. Principe des moindres puissances, $2k$-ièmes et principe des moindres carrés dans les problèmes d'approximation. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 396–399 (1940). [MF 3607]

Let $f(x)$ be a given function on a finite interval (a, b), $k>1$ a positive integer and $p_n(x)$ the approximating polynomial of the nth degree at most characterized by the condition that the integral over (a, b) of the $(2k)$th power of the difference between $f(x)$ and $p_n(x)$ shall be a minimum. A convergent process is indicated for obtaining $p_n(x)$ as limit of a sequence of polynomials defined by means of auxiliary least-square criteria. Further applications of the method are briefly suggested. *D. Jackson.*

Chowla, S. and Auluck, F. C. On Weierstrass approximation theorem. Math. Student **8**, 78–79 (1940). [MF 3284]

An upper bound is obtained for the difference between a function satisfying a general Lipschitz condition and the approximating polynomial given by Landau's integral [Rend. Circ. Mat. Palermo **25**, 337–345 (1908)]. The estimate obtained for the order of magnitude of the error is however not the best possible one for this method of approximation; and a still higher order of approximation is given by other approximating polynomials [cf., e.g., D. Jackson, The Theory of Approximation, New York, 1930, Chapter I]. *D. Jackson* (Minneapolis, Minn.).

Korovkin, P. P. A theorem on series of polynomials. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **6**, 69–70 (1939). (Russian) [MF 3290]

The author applies a theorem of Pólya [S.-B. Preuss. Akad. Wiss., Phys.-Math. Kl. **1928**, 280–282] to show that a series $\sum C_nP_n(z)$, where $P_n(z)$ is a polynomial of degree n having the coefficient of z^n equal to unity, and $\limsup_{n\to\infty}|C_n|^{1/n}=1/r$, cannot converge in a set of measure greater than πr^2. Hence some series (generalized Abel series) discussed by Gontcharoff [Ann. Sci. École Norm. (3) **47**, 1–78 (1930)], and shown by him to converge in a circle [op. cit., p. 13], diverge almost everywhere outside the circle. *R. P. Boas, Jr.* (Durham, N. C.).

Korovkin, P. Expression asymptotique des polynômes orthogonaux sur un contour rectifiable. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 531–534 (1940). [MF 3223]

Let C be a Jordan curve, D the open, $\bar{D}$ the closed exterior of C. Let $f(x)$ be analytic in D, continuous in $\bar{D}$, $f(x)\neq 0$, $f(\infty)>0$. The author deals with the asymptotic expression of the orthogonal polynomials $p_n(x)$ (as $n\to\infty$) satisfying the condition

$$\int_C |f(x)|^2|\delta'(x)|\,p_m(x)\overline{p_n(x)}\,d\sigma=\epsilon_{mn};$$

here $z=\delta(x)$ is the map function of D onto $|z|>1$ in the usual normalization and $d\sigma$ is the arc element of C. The well-known formula

$$p_n(x)\{\delta(x)\}^{-n}\leqq(2\pi)^{-1}\{f(x)\}^{-1},\quad x\in D;\ n\to\infty,$$

is obtained for various sets of conditions imposed on C (or $\delta(x)$) and $f(x)$. Under proper conditions the validity of this formula can be extended into the interior of C [cf. the reviewer's "Orthogonal Polynomials," 1939, p. 365]. *G. Szegö* (Stanford University, Calif.).

Koehler, Fulton. Systems of orthogonal polynomials on certain algebraic curves. Bull. Amer. Math. Soc. **46**, 345–351 (1940). [MF 1846]

Let C be the square in the (x, y)-plane bounded by $x=\pm 1$, $y=\pm 1$ and let $\rho(x, y)=\rho(y, x)=\rho(-x, -y)$ be a weight function on the boundary c of C. Orthogonalization of the sequence $1, x+y, x-y, xy, x^2+y^2, x^2-y^2, x^2y+xy^2, x^2y-xy^2, \cdots$ on c furnishes a sequence of polynomials $p_{mn}(x, y)$ which can be expressed in terms of the polynomials $P_n(x)$ associated with the weight function $\rho(x, 1)$ in $-1\leqq x\leqq+1$. Assuming certain properties of $P_n(x)$, analogous conclusions can be made about $p_{mn}(x, y)$. In particular a convergence and an approximation theorem of this type are discussed. *G. Szegö* (Stanford University, Calif.).

Mazza, S. C. On Hermite polynomials in two variables. An. Soc. Ci. Argentina **130**, 137–148 (1940). (Spanish) [MF 3778]

The author considers Hermite polynomials $H_{m,n}(x, y)$ corresponding to a weight function $\exp[-\frac{1}{2}\psi(x, y)]$, where $\psi(x, y)$ is a positive definite quadratic form. He introduces the adjoint polynomials $G_{m,n}(x, y)$ which form a bi-orthogonal system with $H_{m,n}(x, y)$. He also discusses polynomials $\Theta_{m,n}(x, y)$ which form an orthogonal system with the same weight function ($H_{m,n}$ and $H_{r,s}$ are orthogonal if $m+n\neq r+s$ but not if merely $(m, n)\neq(r, s)$). All three polynomials are expressed in terms of Hermite polynomials $H_n(t)$ of one variable. *E. Hille* (New Haven, Conn.).

Geronimus, J. Generalized orthogonal polynomials and the Christoffel-Darboux formula. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 847–849 (1940). [MF 3577]

The author continues the study of polynomials $\{P_n(z)\}$ orthogonal relative to a given sequence $\{C_n\}$, $n=0, 1, 2, \cdots$

[Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 215–228 (1940); these Rev. **2**, 97]. A recurrence relation is derived for $P_n(z)$ involving $\bar{P}_n(z^{-1})$, which is shown to be characteristic for $P_n(z)$, that is, its validity implies the orthogonality of $P_n(z)$ relative to a properly constructed sequence $\{C_n\}$; it further implies the classical Darboux Formula. The author also discusses the case of a "positive" sequence $\{C_n\}$ (where the known determinants, formed by means of the C_n in constructing $\{P_n(z)\}$, are assumed to be positive). *J. A. Shohat.*

Geronimus, J. Sur quelques propriétés des polynômes orthogonaux généralisés. C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 5–8 (1940). [MF 3694]

The author continues the study of the generalized orthogonal polynomials $P_n(z)$ [cf. the preceding review]. A recurrence relation, of second order, is given for $P_n(z)$, also the relation of $\{P_n(z)\}$ to algebraic continued fractions. It is shown that $\{P_n'(z)\}$ are of the same nature as $\{P_n(z)\}$. The general considerations are applied to the case where the $\{C_n\}$ are "moments" of properly specified weight-functions. *J. A. Shohat* (Philadelphia, Pa.).

Rémès, E. Sur certaines classes de fonctionnelles linéaires dans les espaces C_p et sur les termes complémentaires des formules d'analyse approximative. Acad. Sci. RSS Ukraine. Rec. Trav. [Zbirnik Prace] Inst. Math. **1940**, no. 4, 47–82 (1940). (Ukrainian. Russian and French summaries) [MF 3231]

Many formulae of interpolation, mechanical quadratures, etc. have the character of linear functional operators in the space $C_m(m \geqq 0)$, and so are their remainder terms. The author uses this point of view systematically and derives workable expressions for the remainder terms in various approximate formulae, like interpolation by means of divided differences, Tchebycheff formula of mechanical quadratures, Laplace formula (with interior ordinates), formulae for numerical differentiation, etc. It is assumed that the remainder vanishes identically for polynomials up to a certain degree or certain linear aggregates of properly specified functions. The main point is the integral representation of the remainder, where the integrand is of a suitable form; its sign in various parts of the interval under consideration plays an important part in the discussion. *J. A. Shohat* (Philadelphia, Pa.).

Rémès, E. J. Sur les termes complémentaires de certaines formules d'analyse approximative. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 129–133 (1940). [MF 3546]

This note outlines briefly considerations and results elaborated in the paper reviewed above. *J. A. Shohat.*

Lozinski, S. Über einen Satz von H. Hahn. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **10**, 84–99 (1940). (Russian. German summary) [MF 3304]

The author first proves the sufficiency of the conditions established in the fundamental theorem of Hahn [Math. Z. 1 (1918)] concerning the convergence (I) $\lim_{n\to\infty} p^{(n)}(\bar{x};f) = f(\bar{x})$ of a general continuous distributive interpolation process, for any $f(x)$ of bounded variation on (a, b), at any point of continuity [the author claims, without further explanation, that at this point Hahn's proof is wrong]. The interpolation points are assumed to be arranged in ascending order on the interval (a, b) under consideration. The proof is based upon representing a function of bounded variation as a sum of a continuous function and a function of its saltus, and representing a sequence of discrete sums as a sequence of properly chosen Stieltjes integrals and passing to the limit. He next proves two theorems concerning the convergence (I), provided: (i) the discontinuities of $f(x)$, bounded on (a, b), form at most a countable set; (ii) $f(x)$ is R-integrable on (a, b). The statements of these theorems, as well as their proofs, follow the lines drawn by Pólya in his paper on mechanical quadratures [Math. Z. **37**, 264–286 (1933)]. *J. A. Shohat* (Philadelphia, Pa.).

Lozinski, S. M. Über Interpolation. Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 57–68 (1940). (German. Russian summary) [MF 3188]

Let $f(x)$ be an L-integrable function of period 2π and let $Q_n[f] = Q_n(x;f)$ be the interpolating trigonometric polynomial of degree not greater than n which at the points $x_k = x_k^{(n)} = 2\pi k/(2n+1)$ $(k=0, 1, \cdots, 2n)$ takes on the values

$$\frac{2n+1}{2\pi}\int_{x_{k-1}}^{x_k} f(u)du.$$

Let $\bar{Q}_n$ denote the polynomial conjugate to Q_n and let $\bar{f}$ denote the function conjugate to f. It is shown that a number of results known to be true for the partial sums of Fourier series may be extended to the polynomials Q_n. For example, if $f \in L^p$, where $p>1$, then

$$\lim_{n\to\infty}\int_0^{2\pi} |f(x)-Q_n(x;f)|^p dx = 0,$$

$$\lim_{n\to\infty}\int_0^{2\pi} |\bar{f}(x)-\bar{Q}_n(x;f)|^p dx = 0;$$

if $|f| \log^+ |f|$ is integrable, then

$$\lim_{n\to\infty}\int_0^{2\pi} |f(x)-Q_n(x;f)|\,dx = 0,$$

$$\lim_{n\to\infty}\int_0^{2\pi} |\bar{f}(x)-\bar{Q}_n(x;f)|\,dx = 0;$$

and, finally, if $f \in L$, then

$$\lim_{n\to\infty}\int_0^{2\pi} |f(x)-Q_n(x;f)|^\mu dx = 0,$$

$$\lim_{n\to\infty}\int_0^{2\pi} |\bar{f}(x)-\bar{Q}_n(x;f)|^\mu dx = 0,$$

where μ is any positive number less than 1. There are extensions to the case when the function f is representable by a power series. It may be added that the polynomials Q_n have already been investigated by Offord [Duke Math. J. **6**, 505–510 (1940); these Rev. **1**, 328] for similar purposes. *A. Zygmund* (South Hadley, Mass.).

Lozinski, S. Sur la convergence forte des procédés d'interpolation. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 203–206 (1940). [MF 3601]

It is well known that, if the function $f(x)$ is continuous and of period 2π, then

$$(*) \qquad \lim_{n\to\infty}\int_0^{2\pi} |f(x)-U_n(x)|^p dx = 0$$

for every $p>0$, where $U_n(f) = U_n(x;f)$ is the trigonometric polynomial of degree not greater than n, assuming at the points $2k\pi/(2n+1)$ $(k=0, 1, \cdots, 2n)$ the same values as the

function $f(x)$ [see Marcinkiewicz, Studia Math. 6, 1–17 (1936); Erdös and Feldheim, C. R. Acad. Sci. Paris 203, 913 (1936)]. The author now proves that the result holds for functions f bounded and integrable R.

It is also shown that, if $f(x)$ is bounded and integrable R, and if $g(x)$ is a function such that $|g|\log^+|g|$ is summable, then

$$(**) \qquad \int_0^{2\pi} U_n(f)g\,dx \to \int_0^{2\pi} fg\,dx \qquad \text{as } n\to\infty.$$

If $g(x)$ is summable only, the left hand side of (**) may be unbounded, even for continuous f. It may also be added that there exists a result stronger than (*), namely

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\exp\lambda|f-U_n(f)|\,dx=1$$

for every $\lambda>0$. For f continuous the result is contained implicitly in Marcinkiewicz and Zygmund, Fund. Math. 28, 131–166 (1937), and it may be extended to f bounded and integrable R. *A. Zygmund* (South Hadley, Mass.).

Kuzmin, R. O. and Natanson, I. P. On strong convergence of Lagrange polynomial of interpolation. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. 6, 81–89 (1939). (Russian) [MF 3292]

Let $\omega_0(x), \omega_1(x), \cdots, \omega_n(x), \cdots$ denote the system of orthogonal polynomials corresponding to the weight function $p(x)$ and the interval $-1\leqq x\leqq+1$. Suppose that $x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)}$ are the roots of the equation

$$\omega_n(x)+A_n\omega_{n-1}(x)+B_n\omega_{n-2}(x), \quad A_n, B_n \text{ constants}; B_n\leqq 0.$$

It is assumed that all the roots $x_i^{(n)}$ are situated in the interval $-1\leqq x\leqq+1$. Erdös and Turan showed [Ann. of Math. (2) 38, 142–155 (1937)] that, if $L_n(x;f)$ denotes the Lagrange interpolating polynomial corresponding to the nodal points $x_i^{(n)}$, then

$$(*) \qquad \lim_{n\to\infty}\int_{-1}^{+1} p(x)[f(x)-L_n(x;f)]^2dx=0$$

for any continuous function f. The present paper contains a simple proof of the special case $B=0$ of that theorem. Moreover it is shown on an example that in the case

$$\omega_n(x)=\frac{\sin(n+1)\varphi}{2^n\sin\varphi}, \qquad x=\cos\varphi,$$

the relation (*) ceases to be true, if the factor $p(x)$ is dropped from the integrand. *A. Zygmund.*

Webster, M. S. Maximum of certain fundamental Lagrange interpolation polynomials. Bull. Amer. Math. Soc. 47, 71–73 (1941). [MF 3812]

The author investigates the maxima of the fundamental functions $|l_k^{(n)}(x)|$ in the case of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. Among others he proves that if $\alpha, \beta<3/2$ and $-1+\epsilon<x_k<1-\epsilon$ then $\max|l_k^{(n)}(x)|\to 1$. *P. Erdös.*

Kharchiladzé, Philippe. Sur l'interpolation trigonométrique. Rec. Math. [Mat. Sbornik] N.S. 8 (50), 471–487 (1940). (Russian. French summary) [MF 3744]

Let $U_n(x,f)$ denote the trigonometric polynomial of order not greater than n coinciding with the function $f(x)$ at the $2n+1$ points $x_i=2\pi i/(2n+1)$ $(n=0, 1, \cdots, 2n)$, and let $U_{n,\nu}(x,f)$ denote the νth partial sum of $U_n(x,f)$. Bernstein proved [C. R. (Doklady) Acad. Sci. URSS (N.S.) 2, 1–8 (1934)] the following analogue of the classical theorem of Fejér: if $f(x)$ is continuous and of period 2π, the expression

$$\sigma_{n,\nu}=\frac{1}{\nu+1}\sum_{\mu=0}^{\nu}U_{n,\mu}(x,f)$$

tends uniformly to $f(x)$ for $n\geqq\nu\to\infty$. The author now applies this result to simplify proofs of the following known results: (i) if $f(x)$ is continuous, periodic and of bounded variation in $(0, 2\pi)$, $U_n(x,f)$ tends uniformly to $f(x)$; (ii) if $f(x)$ is continuous, and if $U^*(x,f)=\limsup_{n\to\infty}U_n(x,f)<+\infty$ for every point x of a set E of positive measure, then

$$U_*(x,f)=\liminf_{n\to\infty}U_n(x,f)>-\infty,$$

$$2f(x)=U^*(x,f)+U_*(x,f)$$

almost everywhere in E [Marcinkiewicz, Studia Math. 6, 1–15 (1936)]; (iii) if $f(x)$ is continuous and periodic, then

$$\tfrac{1}{2}\left\{U_n\left(x+\frac{\pi}{2n+1},f\right)+U_n\left(x-\frac{\pi}{2n+1},f\right)\right\}$$

tends uniformly to $f(x)$ [Bernstein, loc. cit.]. Moreover, it is shown that, if f is integrable L, and

$$F(x)=\int_0^x f(t)dt, \quad F(0)=F(2\pi)=0,$$

the polynomials $dU_n(x,F)/dx$ possess certain properties analogous to the properties of the partial sums of the Fourier series of $f(x)$ [see also Marcinkiewicz, loc. cit.]. *A. Zygmund* (South Hadley, Mass.).

Daniell, P. J. Remainders in interpolation and quadrature formulae. Math. Gaz. 24, 238–244 (1940). [MF 3360]

The author seeks to derive the remainder in various interpolation and quadrature formulae for differentiable functions, in case they are "of order n," that is, $L(x^k)=0$ (L-remainder which is called, rather confusingly, "formula"), $k=0, 1, \cdots, n-1$, $L(x^n)=E\neq 0$, and are "simplex," that is, $L(f)=0$ implies $f^{(n)}(\xi)=0$, ξ belonging to the interval under consideration. The reviewer feels that many statements, particularly those dealing with E and the notion "simplex," need proofs or, at least, elucidation, as, for example, the statement that the determination of E "is easy," or that deciding whether a formula is or is not simplex ". . . is always possible by expressing $L(f)$ as an integral in terms of $f^{(n)}(x)$ . . ." [cf. the Lagrange interpolation formula!]. *J. A. Shohat* (Philadelphia, Pa.).

Besicovitch, J. S. Process of mechanical quadratures for improper integrals. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. 6, 36–42 (1939). (Russian) [MF 3286]

The author applies the Gaussian formula of mechanical quadratures

$$(1) \qquad \int_{-1}^{1} f(x)p(x)dx\sim\sum_{i=1}^{n}A_{i,n}f(x_{i,n}), \quad n=1, 2, \cdots,$$

where the $x_{i,n}$ are the zeros of the orthogonal polynomials $\varphi_n(x;-1,1;p(x))$ with $p(x)=(1-x^2)^{-\frac{1}{2}}$, $(1-x^2)^{\frac{1}{2}}$, $(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$, to certain classes of improper integrals, namely, where $f(x)p(x)$ is, say, $O((1-x)^{\alpha-1})$, $0<\alpha<1$, near $x=1$. Making use of the known explicit expressions of $A_{i,n}, x_{i,n}$, for these $p(x)$, the usual method is employed, that is, estimating the contribution of the part of the summation $\sum_{i=1}^{n}A_{i,n}x_{i,n}$ furnished by the $x_{i,n}$ near $x=1$. In this way

the convergence of (1), as $n\to\infty$, is established for the improper integrals under discussion. *J. A. Shohat.*

Achyeser, N. I. and Krein, M. G. On some quadrature formulas of P. Tschebysheff and A. Markoff. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 15–28. (Russian) [MF 3498]

The function $q(x)$ defined in an interval (a, b), $-\infty \leqq a < b \leqq \infty$ is said to admit of a quadrature formula of Markoff of order $(2n-2)$ if the integrals $c_i=\int_a^b x^i q(x)dx$, $i=0, 1, 2, \cdots, 2n-2$, $c_0=0$, exist and, for a suitable choice of constants α_i, β_i, $a\leqq\beta_1\leqq\alpha_1\leqq\cdots\leqq\beta_{n-1}\leqq\alpha_{n-1}\leqq b$ and L, the formula

$$\int_a^b P_{2n-2}(x)q(x)dx=L\sum_{i=1}^{n-1}\{P_{2n-2}(\alpha_i)-P_{2n-2}(\beta_i)\}$$

holds for an arbitrary polynomial of degree not greater than $(2n-2)$. The function $q(x)$ is said to admit a quadrature formula of Tschebysheff of order $(2n-1)$ if the integral c_{2n-1} also exists and if, for a suitable choice of constants α_i, β_i, $a\leqq\alpha_1\leqq\cdots\leqq\alpha_{n-1}\leqq b$, $a\leqq\beta_1\leqq\cdots\leqq\beta_{n-1}\leqq b$, and K, the formula

$$\int_a^b P_{2n-1}(x)q(x)dx=K\sum_{i=1}^{n-1}\{P(\alpha_i)-P(\beta_i)\}$$

holds for an arbitrary polynomial of degree not greater than $(2n-1)$. The authors prove that a necessary condition that a function $q(x)$, for which

$$c_0=\int_{-\infty}^{\infty}q(x)dx,\quad c_{k+1}\equiv(k+1)s_k=\int_{-\infty}^{\infty}x^{k+1}q(x)dx,$$

$$k=0, 1, 2, \cdots, 2n-1,$$

exist, should admit a quadrature formula of Markoff of order $2n$ is that the quadratic form $\sum_{i,k=1}^{n-1}s_{i+k}x_ix_k$ be definite (positive or negative). Assuming that this form is positive definite the authors obtain also conditions in order that $q(x)$ should admit of a quadrature formula of Tschebysheff. *J. D. Tamarkin* (Providence, R. I.).

***Shohat, J. A., Hille, Einar and Walsh, Joseph L. A Bibliography on Orthogonal Polynomials.** Bull. Nat. Research Council, no. 103. Published by the National Research Council of the National Academy of Sciences, Washington, D. C., 1940. ix+204 pp. $3.00.

The present bibliography means much more than a practically complete collection of references. By use of the code symbol system it furnishes also an indication of the content and in many cases of the results of the individual papers. Those interested in the theory of orthogonal polynomials will doubtless appreciate the publication of this highly useful book and they will be certainly aware of the immense amount of work invested in this endeavor. *G. Szegö.*

Differential Equations

Shiefner, L. On the integration of some differential systems in finite form. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 341–348 (1940). (Russian. English summary) [MF 3326]

This note obtains the solution of the system of differential equations $dY/dx=Y[U_1\cdot\phi_1(x)-U_2\cdot\phi_2(x)]$ by means of quadratures, Y, U_1 and U_2 being n-rowed square matrices satisfying the conditions $[U_1[U_2U_1]]=0$ and $[U_2[U_2U_1]]=0$. Most of the paper is concerned with the study of matrices satisfying the last two commutator conditions. If the characteristic roots of U_1 are $\xi_1, \cdots, \xi_n$, the most general matrix U_2 is obtained in terms of these roots.

M. S. Knebelman (Pullman, Wash.).

Pompeiu, D. Remarques sur l'équation de Riccati. C. R. Acad. Sci. Paris **210**, 692–693 (1940). [MF 3617]

Soit l'équation de Riccati $y'=py^2+qy+r$, où p, q, r sont des fonctions continues et dérivables de x. Si $y=f(x)$ est une intégrale de cette équation, et $k(x)=-q/2p$, $\varphi(x)=2k(x)-f(x)$, alors $\varphi(x)$ satisfait à l'équation associée $\varphi'=p_1\varphi^2+q_1\varphi+r_1$, avec $p_1=-p$, $q_1=4pk+q$. L'intégration de ces deux équations s'entraîne mutuellement. *E. Feldheim.*

Kojalovich, B. M. On the question of integration of the differential equation $ydy-ydx=R(x)dx$. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 79–87. (Russian) [MF 3506]

The author remarks: "Questions of formal integration do not at present attract the attention of scholars. One even hears disparaging remarks about 'mastery of quadrature' . . . I can not help but feel that the source of this is the same as in the fable of the fox and the grapes." It is shown that, if 3, 4, 5 or 6 particular solutions are known, they must satisfy certain (differential) conditions and in the last three cases the complete primitive may be obtained by quadratures. *M. S. Knebelman* (Pullman, Wash.).

Ritt, J. F. On a type of algebraic differential manifold. Trans. Amer. Math. Soc. **48**, 542–552 (1940). [MF 3171]

Le système des solutions d'un système de polynômes differentiels dans une seule fonction inconnue est appelé limité quand la fonction 0 n'est pas une limite des réciproques des fonctions du système. Somme et produit de deux systèmes sont les systèmes minimes contenant respectivement toutes les sommes et tous les produits de deux fonctions appartenant aux deux systèmes donnés; dérivé d'un système est le système minime contenant les dérivées des fonctions de ce système. La somme et le produit de deux systèmes limités est limité si ces systèmes sont du premier ordre; peuvent ne pas l'être si l'ordre est supérieur. Le dérivé d'un système limité est toujours limité.

B. Levi (Rosario).

Zaremba, Stanislas Christian. À propos des champs de demi-cônes convexes. Bull. Sci. Math. **64**, 5–12 (1940). [MF 1974]

Dans une série de travaux l'auteur et Marchaud avaient étudié des champs continus de cônes convexes et des courbes dont les demi-tangentes en chaque point se trouvent sur le cône en ce point. Ce travail contient quelques remarques sur les changements qu'il faut faire si l'on remplace le champ continu de cônes par un champ de pinceaux de droites semi-continu supérieurement. Les formulations des résultats sont trop compliquées pour être reproduites.

P. Scherk (New York, N. Y.).

Rosenblatt, Alfred. Sur l'unicité des solutions des équations différentielles ordinaires. Actas Acad. Ci. Lima **3**, 94–98 (1940). [MF 3269]

The second order differential system $y'=f(x, y, z)$, $z'=g(x, y, z)$, where $f(x, y, z)$ and $g(x, y, z)$ are continuous

on $0 \leqq x \leqq a$, $|y| \leqq H$, $|z| \leqq H$, is considered. Under the hypotheses

$$|f(x, y_1, z_1) - f(x, y_2, z_2)| \leqq 2|z_1 - z_2|/x,$$
$$|g(x, y_1, z_1) - g(x, y_2, z_2)| \leqq k|y_1 - y_2|, \quad 0 < k < 1,$$

the system is shown to have a unique solution which vanishes at $x=0$. It is pointed out that the same conclusions are valid when the right hand sides of the inequalities are replaced by $A|z_1-z_2|/x^\alpha$, $\alpha<2$, and $k|y_1-y_2|$, respectively. It is also pointed out that these conclusions would follow if the right hand sides of these inequalities were replaced by $A|y_1-y_2|+B|z_1-z_2|/x$, $C|y_1-y_2|/x$, where $BC=1$.

W. M. Whyburn (Los Angeles, Calif.).

Jelchin, M. Sur une méthode d'évaluation de la phase d'une équation différentielle linéaire du second ordre. Uchenye Zapiski Moskov. Gos. Univ. Matematika **45**, 97–108 (1940). (Russian. French summary) [MF 3735]

The author points out that the transformation

$$y = z \exp\left[-\frac{1}{2}\int_{t_0}^{t} p(\xi)d\xi\right],$$

applied to the differential equation (1) $\ddot{y}+p(t)\dot{y}+q(t)=0$, will yield (2) $\ddot{z}=I(t)z$, where $I(t)=\frac{1}{2}\dot{p}+\frac{1}{4}p^2-q$ is an invariant of (1). The solution of (2) can be given in the form

$$z = R_0\omega^{-\frac{1}{2}}(t)\cos\sigma(t), \quad \sigma(t) = \int_{t_0}^{t}\omega(\xi)d\xi + \gamma_0,$$

where R_0, γ_0 are constants and where $\omega(t)$ (variable "frequency") satisfies

$$-\tfrac{1}{2}(\omega'/\omega)' + \tfrac{1}{4}(\omega'/\omega)^2 - \omega^2 = I(t);$$

$\sigma(t)$ is termed the "phase" of the solution [Hamel, Math. Ann. **73** (1913)]. The author previously proved [C. R. (Doklady) Acad. Sci. URSS, no. 3 (1938)] that the phase increases when I decreases. In the present paper the author gives an effective process of approximate evaluation of $\sigma(t)$ on a bounded interval (t_0, t_1). An application yields a sufficient condition for Jacobi's condition in the variational problem related to $\int_a^b F(x, y, y')dx$. Another application, when $I(t)$ is periodic, generalizes for the equation (2) the results of Liapounoff in his classical memoir relating to stability of motion. *W. J. Trjitzinsky* (Princeton, N. J.).

Shin, D. On solutions of the system of quasi-differential equations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 391–395 (1940). [MF 3606]

The author briefly outlines an investigation, closely similar to that given by him in Rec. Math. [Mat. Sbornik] N.S. **7** (**49**), 479–532 (1940) [these Rev. **2**, 51]. In the present contribution the equations are given in vector form.

W. J. Trjitzinsky (Princeton, N. J.).

Cinquini, Silvio. Un'osservazione sopra i problemi di valori al contorno per l'equazione $y''=f(x, y, y')$. Boll. Un. Mat. Ital. (2) **2**, 322–325 (1940). [MF 2984]

Dans une note précédente [Ann. Scuola Norm. Super. Pisa (2) **8**, 271–283 (1939); these Rev. **2**, 51] l'auteur avait démontré l'existence d'une solution de l'équation $y''=f(x, y, y')$ passant pour (x_0, y_0), (x_1, y_1) en supposant vérifiée, entre autres conditions, une limitation de la forme $|f(x, y, y')| \leqq \gamma(y)\phi(y')+\psi_1(x)$, où $\phi(u) \geqq k|u|$ (k const. >0); on observe ici que le second membre de cette limitation peut être généralisé en $\gamma(y)\phi(y')+\psi_1(x)\phi_1(y')$, où l'on suppose, en forme alternative, ou $\psi_1(x)$ presque partout nulle ou bien $\phi_1(u) \geqq 0$, $\phi(u) \geqq k|u|\phi_1(u)$. *B. Levi* (Rosario).

Mikeladze, Sch. E. Über die Lösung von Randwertproblemen mit der Differenzenmethode. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 400–402 (1940). [MF 3608]

The author applies the method of finite differences to approximate the eigenvalues of an ordinary linear differential equation with a linear parameter under vanishing boundary conditions. *A. E. Heins.*

Finkelstein, G. M. On the structure of the Green function of an ordinary differential operator. Acad. Sci. RSS Ukraine. Rec. Trav. [Zbirnik Prace] Inst. Math. **1940**, no. 4, 165–174 (1940). (Ukrainian. Russian and English summaries) [MF 3235]

On writing

$$L = \sum_0^n l_k y^{(k)}, \quad L' = \sum_0^n (-1)^k \frac{d^k}{dx^k}(l_k y)$$

(l_k having k continuous derivatives for $a \leqq x \leqq b$), corresponding to boundary conditions of Sturmian type there is a kernel $K(x, s) = \sum_1^q \psi_i(x)\chi_i(s)$ $(x \leqq s)$ and $= \sum_{q+1}^n \psi_i(x)\chi_i(s)$ $(x \geqq s)$, where the ψ_i and χ_i are certain full sets of solutions for $L=0$ and $L'=0$, respectively. Let $K^{(p,q)}(x, s)$ $(p+q=n)$ be so formed, corresponding to conditions $y^{(j)}(a)=0$ $(j \leqq p-1)$, $y^{(i)}(b)=0$ $(i \leqq q-1)$, non-singular for L. The author establishes a rather simple relation between the ψ_i and χ_j, with the aid of which a new set of necessary and sufficient conditions is obtained under which one of the kernels $\pm K^{(p,q)}$ is such that, denoting it by H, one has

$$H\begin{pmatrix} x_1, \cdots, x_m \\ s_1, \cdots, s_m \end{pmatrix} \geqq 0, \quad \text{for } a \leqq x_j;\ s_j \leqq b;\ x_{j-1} < x_j;\ s_{j-1} < s_j,$$

$$H\begin{pmatrix} x_1, \cdots, x_m \\ x_1, \cdots, x_m \end{pmatrix} > 0, \quad \text{for } a < x_1 < \cdots < x_m < b,$$

for $m = 1, 2, \cdots$. *W. J. Trjitzinsky* (Princeton, N. J.).

Kalafaty, P. Sur les fonctions de Green des équations différentielles ordinaires. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 526–530 (1940). [MF 3559]

The work of this paper is closely related to that reported by M. Krein and G. Finkelstein in papers that appeared in this same journal for 1939 [pp. 220–223, 643–646; see these Rev. **2**, 52]. The paper studies the differential equation

$$L(y) \equiv \rho_0 \frac{d}{dx}\rho_1 \frac{d}{dx}\rho_2 \frac{d}{dx} \cdots \rho_{n-1}\frac{d}{dx}\rho_n y = 0,$$

where the coefficients are positive real functions, together with linear boundary conditions at the two end points of the interval of definition. It establishes further theorems of the type developed in the two papers cited above. One of the theorems established states that if the Green's function for the differential system is a kernel of Kellogg [see Kellogg, Amer. J. Math. **40**, 147 (1918)] then the boundary conditions are those of Sturm. *W. M. Whyburn.*

Levine, Jack. Invariants of systems of second order linear differential equations. Duke Math. J. **7**, 298–311 (1940). [MF 3403]

The system

$$d^2y^i/dx^2 + L_j^i(x)dy^j/dx + M_j^i(x)y^j = 0$$

is subjected to the mapping $\bar{y}^i = a_j^i(x)y^j$. $\bar{L}$ and $\bar{M}$ are obtained from the transform of the system. A seminvariant of order r of this system is a function f of L, M, dL/dx and their derivatives of orders $1, 2, \cdots, r$ which retains its form under the above mapping. The system is first reduced to semi-canonical form in which f is a function of $M, \cdots, d^rM/dx^r$ (replacement theorem) and by methods analogous to those used for affine differential invariants a fundamental set of seminvariants is constructed. The author gives also a construction for semi-covariants $(f(y, dy/dx, L, M, \cdots))$ and of invariants, that is, seminvariants which retain their form under an arbitrary transformation $\bar{x} = \bar{x}(x)$.

M. S. Knebelman (Pullman, Wash.).

Halpern, S. Quelques remarques sur les systèmes aux différentielles totales. Uchenye Zapiski Moskov. Gos. Univ. Matematika **45**, 93–96 (1940). (Russian. French summary) [MF 3734]

This note deals with the complete integrability of the system

$$L_\alpha(w^i) \equiv \frac{\partial w^i}{\partial t^\alpha} - \sum a_{j\alpha}^i(t)w^j = F_\alpha{}^i(t), \quad i = 1, \cdots, p; \ \alpha = 1, 2.$$

[The notation is the reviewer's.] The functions $F_\alpha{}^i$ and the square matrices $A_\alpha \equiv \|a_{j\alpha}^i\|$ are assumed to have continuous first partial derivatives. The author proves the classical theorem (by classical methods) that the usual conditions of integrability,

$$A_\alpha A_\beta - A_\beta A_\alpha = \frac{\partial A_\alpha}{\partial t^\beta} - \frac{\partial A_\beta}{\partial t^\alpha}$$

and $L_\alpha(F_\beta{}^i) - L_\beta(F_\alpha{}^i) = 0$, are necessary and sufficient for complete integrability. *M. S. Knebelman.*

González, Mario O. Differential equations of the first order invariant with respect to the group $Uf = \xi(x, y)(\partial f/\partial x) + \eta(x, y)(\partial f/\partial y)$, where ξ and η satisfy the condition $\partial\xi/\partial x = -\partial\eta/\partial y$. Bol. Mat. **13**, 158–162 (1940). (Spanish) [MF 2704]

Pâquet, P.-V. Sur les invariants paramétriques d'ordre c d'une variété n-uple dans un espace à N dimensions. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **25**, 388–400 (1939). [MF 2544]

Let

$$x^\mu = x^\mu(t^1, t^2, \cdots, t^n), \quad \mu = 1, 2, \cdots, N = n+m,$$

with $(t^1, t^2, \cdots, t^n)$ in a domain D, be the parametric equations of a regular differentiable n-dimensional variety $\Sigma_n{}^N$ immersed in an N-dimensional number space. By the classical Lie methods of infinite continuous transformation groups, the author obtains the complete system of first order partial differential equations that is satisfied by (absolute scalar) differential invariants of order c of $\Sigma_n{}^N$. The author first shows that there exists a fundamental set of $n+m\binom{n+c}{c}$ (functionally independent) differential invariants of order c (at most) of $\Sigma_n{}^N$ under arbitrary bidifferentiable topological transformations of D into a domain $\bar{D}$. The author then exhibits a set S of $n+m\binom{n+c}{c}$ differential expressions and shows that S is locally a fundamental set of differential invariants. The subject matter of this paper is closely related to the studies on parametric multiple integral invariants of the calculus of variations. *A. D. Michal.*

Pfeiffer, G. Sur la représentation spéciale d'une équation linéaire en Jacobiens aux derivées partielles du premier ordre à plusieurs fonctions inconnues, satisfaisant aux conditions de M. Hamburger, et d'un système d'équations linéaires en Jacobiens, satisfaisant aux conditions de M. Hamburger généralisées. Acad. Sci. RSS Ukraine. Rec. Trav. [Zbirnik Prace] Inst. Math. **1940**, no. 5, 17–30 (1940). (Ukrainian. Russian and French summaries) [MF 3749]

Pfeiffer, G. Passage de la méthode générale de Jacobi d'intégration des systèmes complets d'équations non linéaires à la méthode simplifiée. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 99–101 (1940). [MF 3596]

The proof of the transition expressed in the title of this paper is taken for granted in the books by Goursat [1891, p. 155] and Forsyth [1906, vol. 5, p. 122]. The author attempts to improve on the explanations given in Goursat's "Leçons" of 1921, p. 75, 278. He sees the reasons for the transition in the fact that the system of linear equations in question has integrals which are linear in $p_1, p_2, \cdots, p_n$ with constant coefficients. This point of view is expressed in two theorems, from which the results of Goursat and Forsyth can be deduced. A third theorem expresses a generalization, due to work by Gilbert [Ann. Soc. Sci. Bruxelles **1** (1881)] and by the author [Acta Math. **61**, 239–261 (1933); Bull. Acad. Sci. USSR [Izvestia] (7) **1932**, 131–151]. *D. J. Struik.*

Pfeiffer, G. La recherche des divers types d'intégrales de Lagrange. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 101–103 (1940). [MF 3210]

Continuation of the paper in the same C. R. **23**, 119–120 (1939). Given are complete systems of m nonlinear partial differential equations of the first order in n variables $x_1, \cdots, x_n$, and a solution is offered by adjunction (a) of a set of $m-n$ equations of the form $F(x_1, \cdots, x_n, p_1, \cdots, p_n) = a$ and (b) of a set of $m-n+1$ equations of the form $G(x_1, \cdots, x_n, z, p_1, \cdots, p_n) = a$. The case in which the equations contain only fractions $p_1/p_n, \cdots, p_{n-1}/p_n$ receives special attention. An example is $pq = 4xy$ with complete integral $z = ax^2/2 + 2y^2/a + b$ and $z = ((x^2+\alpha)(y^2+\beta))^{\frac{1}{2}}$ [see G. Pfeiffer, Bull. Sci. Math. **53**, 2 (1929)]. *D. J. Struik.*

Malkin, I. Sur un théorème d'existence de Poincaré-Liapounoff. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 307–311 (1940). [MF 3242]

The paper is concerned with the system of equations

$$(1) \quad \frac{\partial z_i}{\partial t} + \sum_{s=1}^{n} (p_{s1}x_1 + \cdots + p_{sn}x_n + X_s)\frac{\partial z_i}{\partial x_s} = Z_i(t, x_1, \cdots, x_n, z_1, \cdots, z_k), \quad i = 1, 2, \cdots, k,$$

in which the p_{sj} are real continuous bounded functions of t for $t \geqq 0$, while the X_s and Z_i are analytic in $x_1, \cdots, x_n$, $z_1, \cdots, z_k$, in some region, their developments in powers of these variables having coefficients which are real continuous bounded functions of t, and satisfying the conditions that the developments of the X_i contain no constant or linear terms, and those of the Z_i contain no constant terms and no linear terms in the $z_1, \cdots, z_k$. The following theorems are proved. Theorem II. If there exist real constants a and b, and a form $V(t, x_1, \cdots, x_n)$, quadratic in $x_1, \cdots, x_n$, with coefficients which are bounded functions of t for $t \geqq 0$, such that

(i) $\quad V(t, x_1, \cdots, x_n) \geqq a^2(x_1^2 + \cdots + x_n^2)$,

and

$$\text{(ii)}_2 \quad \frac{\partial V}{\partial t}+\sum_{s=1}^{n}\frac{\partial V}{\partial x_s}(p_{s1}x_1+\cdots+p_{sn}x_n)\geqq b^2(x_1^2+\cdots+x_n^2),$$

then there exists a unique system of real functions $z_i(t, x_1, \cdots, x_n)$ which (α) for $t\geqq 0$ satisfy the equations (1), are analytic in $x_1, \cdots, x_n$ in a certain region and have developments in powers of $x_1, \cdots, x_n$ whose coefficients are bounded functions of t, and which (β) reduce at $t=t_0\geqq 0$ to arbitrarily prescribed analytic functions $\varphi(x_1, \cdots, x_n)$ whose developments contain no constant terms. Theorem I. If the hypothesis of theorem II is modified by replacing (ii)$_2$ by (ii)$_1$

$$\text{(ii)}_1 \quad \frac{\partial V}{\partial t}+\sum_{s=1}^{n}\frac{\partial V}{\partial x_s}(p_{s1}x_1+\cdots+p_{sn}x_n)\leqq -b^2(x_1^2+\cdots+x_n^2),$$

the conclusion (α) follows. *R. E. Langer.*

Thomas, Joseph Miller. Orderly differential systems. Duke Math. J. 7, 249–290 (1940). [MF 3401]

The author gives a new proof of Riquier's theorem on orthonomic systems of partial differential equations, in which he uses a set of inequalities in place of the old idea of cotes [cf. J. M. Thomas, Ann. of Math. (2) **30**, 290 (1929)]. His methods then permit him to prove a similar theorem for orderly systems, that is, a system decomposable into a finite number of orthonomic systems. Orderly systems not only contain orthonomic systems as a proper subclass, but even contain systems which are outside the scope of Riquier's general theory on non-orthonomic systems. The paper also gives a new method for testing whether a given system is orthonomic. This new test based on the above mentioned set of inequalities is simpler than the method based on cotes.

P. 262, line 3 insert after A_1: "or in an equation of the prolonged system whose ordinal is less than that of A_2."

F. G. Dressel (Durham, N. C.).

Schouten, J. A. Ueber Differentialkomitanten zweier kontravarianter Grössen. Nederl. Akad. Wetensch., Proc. **43**, 449–452 (1940). [MF 2255]

Construction of an affinor depending on two contravariant (or covariant) affinors of arbitrary valence and containing first derivatives. For the case that the two affinors $P^{\alpha\varepsilon_1\cdots\varepsilon_a}$ and $Q^{\beta\varepsilon_1\cdots\varepsilon_b}$ are symmetrical, this differential comitant is

$$R^{\varepsilon_1\cdots\varepsilon_{a+b+1}}=(a+1)P^{\beta(\varepsilon_1\cdots\varepsilon_a}\partial_\mu Q^{\varepsilon_{a+1}\cdots\varepsilon_{a+b+1})}-(b+1)Q^{\beta(\varepsilon_1\cdots\varepsilon_b}\partial_\mu P^{\varepsilon_{b+1}\cdots\varepsilon_{a+b+1})}.$$

For the general case we refer to the paper itself. This comitant is related to the Lie differential of an affinor with respect to a certain direction and to the condition for completeness of systems of partial differential equations.

D. J. Struik (Cambridge, Mass.).

Schouten, J. A. und van der Kulk, W. Ueber algebraische Systeme von partiellen Differentialgleichungen erster Ordnung. I. Gleichungen mit einer Unbekannten. Nederl. Akad. Wetensch., Proc. **43**, 955–963 (1940). [MF 3530]

The theory of systems of Pfaff equations [the same Proc. **43**, 18–31, 179–188, 453–462, 674–686 (1940); cf. these Rev. **1**, 145 and **2**, 54] leads to so-called algebraic systems of partial differential equations of the first order, which are systems of which the left hand sides are homogeneous integer rational functions of the coordinates of a vector $w_\lambda=\partial_\lambda s_1+s_2\partial_\lambda s_2+\cdots$, where $s_1, s_2, s_3, \cdots$ are the unknown functions and $\partial_\lambda=\partial/\partial x^\lambda$, $\lambda=1, 2, \cdots, n$. This paper presents the case of only one unknown function, hence $w_\lambda=\partial_\lambda s$. For this purpose N symmetrical tensors $P_i^{\varepsilon\lambda_1\cdots\lambda_{a_i}}$, $i=1, 2, \cdots, N$, are introduced with coordinates analytic in the x^ε. If $\mathfrak{M}$ is the set of all covariant pseudovector fields w_λ satisfying the equations

$$P_i^{\varepsilon\lambda_1\cdots\lambda_{a_i}}w_\mu w_{\lambda_1}\cdots w_{\lambda_{a_i}}=0,$$

then we can consider every field w_λ as a point of an $(n-1)$ dimensional projective space P_{n-1}, if the basic domain is given by the meromorphic functions of x^ε. $\mathfrak{M}$ is an algebraic manifold in P_{n-1}, and is, for $N=1$, a hypersurface of $n-2$ dimensions. The tensor P is said to vanish on $\mathfrak{M}$. The totality of all tensors P which vanish on $\mathfrak{M}$ is the ideal belonging to $\mathfrak{M}$. A manifold is complete if the vanishing of two tensors P and Q implies the vanishing of a certain given differential comitant $D(P; Q)$ [cf. the paper reviewed above]. The paper further deals with theorems on irreducible parts of complete manifolds and on projecting manifolds.

D. J. Struik (Cambridge, Mass.).

Schouten, J. A. und van der Kulk, W. Ueber algebraische Systeme von partiellen Differentialgleichungen erster Ordnung. II. Gleichungen mit einer Unbekannten. Nederl. Akad. Wetensch., Proc. **43**, 1160–1170 (1940). [MF 3726]

This paper culminates in the following existence theorem: Let $\mathfrak{M}$ be a complete irreducible algebraic manifold of d dimensions, $\overset{0}{x}{}^\lambda$ a point where the local algebraic manifold belonging to $\mathfrak{M}$ is also irreducible and let $\overset{0}{w}_\lambda$ be a vector at $\overset{0}{x}{}^\lambda$ belonging to $\mathfrak{M}$. Let $\mathfrak{M}$ have a tangential space at $\overset{0}{x}{}^\lambda$, $\overset{0}{w}_\lambda$. In this case there exists at least one gradient field $\partial_\lambda s$ belonging to $\mathfrak{M}$ and analytical in the neighborhood of $\overset{0}{x}{}^\lambda$, of which the field value at $\overset{0}{x}{}^\lambda$ coincides with $\overset{0}{w}_\lambda$. This existence theorem also possesses an inverse as follows: If at every point $\overset{0}{x}{}^\lambda$ of a definite domain of a X_n every vector $\overset{0}{w}_\lambda$ of a definite domain of the space of $\overset{0}{w}_\lambda$, and belonging to an algebraic manifold $\mathfrak{M}$, can be continued into a gradient field belonging to $\mathfrak{M}$, then $\mathfrak{M}$ is complete. *D. J. Struik.*

Bergman, Stefan. Boundary values of functions satisfying a linear partial differential equation of elliptic type. Proc. Nat. Acad. Sci. U. S. A. **26**, 668–671 (1940). [MF 3148]

Dans deux travaux précédents l'auteur avait donné pour la solution générale d'un certain type d'équations aux dérivées partielles elliptiques du deuxième ordre à coefficients analytiques l'expression

$$U(z, \bar{z})=\int_{-1}^{1}E(z, \bar{z}; t)f(\tfrac{1}{2}z(1-t^2))(1-t^2)^{-\frac{1}{2}}dt,$$

où E est de la forme $E=c+z\bar{z}t^2E^*(z, \bar{z}; t)$ et $f(\zeta)=\sum a_n\zeta^n$ est une fonction analytique qu'on peut choisir arbitrairement; on peut supposer que le rayon de convergence est 1. Il résulte un développement de $U(z, \bar{z})$ en serie convergente à l'intérieur du cercle de rayon 2. Sous la condition que la serie $\sum|a_n|^2n^{-1}$ soit convergente, cette fonction est continue radialement presque partout sur la circonférence. [Il paraît nécessaire d'admettre pour un certain passage la condition légèrement plus restrictive que soit convergente $\sum|a_n|n^{-\frac{1}{2}}$]. Il y a quelque difficulté pour transformer cette condition, qui se rapporte aux coefficients a_n, en une se rapportant aux valeurs au contour de la solution cherchée.

En exprimant la solution au moyen d'une fonction de Green ou d'une solution de l'équation adjointe on peut surmonter en partie cette difficulté. *B. Levi* (Rosario).

Bergman, Stefan. The approximation of functions satisfying a linear partial differential equation. Duke Math. J. 6, 537–561 (1940). [MF 2714]

L'auteur étudie l'approximation de la solution de certains types d'équations linéaires aux dérivées partielles, déterminée par conditions au contour ou par conditions de Cauchy. Précisément il considère l'équation hyperbolique homogène

(1) $G(U)=U_{xy}+A(x,y)U_x+B(x,y)U_y+C(x,y)U=0$

et l'équation elliptique analogue

(2) $L(U)=U+A_1(x,y)U_x+A_2(x,y)U_y+C(x,y)U=0,$

où $A_1+iA_2=A(z)$ est supposé former une fonction de la variable complexe $z=x+iy$, régulière dans le domaine d'integration; cette équation se ramène à (1) par l'introduction des nouvelles variables z et $\bar{z}=x-iy$.

Dans un travail précédent [Rec. Math. [Mat. Sbornik] N.S. 2 (44), 1169–1197 (1937)] l'auteur avait calculé sous forme de série une solution de (1), qu'il appelle fonction génératrice de première espèce, de la forme

$$E(x,y,t)=H(x,y)E^*(x,y,t),$$

avec $\log H=n(x)-\int_0^y A(x,y)dy$, où $E^*(x,y,t)$ est une solution de l'équation aux dérivées partielles

$$-(1-t^2)E^*_{yt}+t^{-1}E^*_y-2tx[E^*_{xy}+DE^*_y+FE^*]=0$$

de la forme

$$E^*=c+txyE_1^*(x,y,t),$$

où

$$D=D(x,y)=n_x-\int_0^y A_x dy+B,\quad F=F(x,y)=-A_x-AB+C;$$

et il avait démontré que la solution générale de (1) peut s'écrire

$$U(x,y)=\int_{-1}^{1}[E_1(x,y,t)f(\tfrac{1}{2}x(1-t^2))+E_2(x,y,t)g(\tfrac{1}{2}y(1-t^2))](1-t^2)^{-\frac{1}{2}}dt,$$

où E_1 et E_2 représentent respectivement la E précédente et la fonction analogue en échangeant x et y, et $f(u)$, $g(u)$ sont deux fonctions analytiques de u. En décomposant l'intégrale dans ses deux termes et posant successivement au lieu de $f(u)$ et de $g(u)$ les puissances u^n, on obtient une double succession de fonctions que sont les solutions fondamentales de l'équation.

En conséquence de l'approximation par polynômes des fonctions f, g, on montre séparément pour le cas elliptique, et après pour le cas hyperbolique, que toute solution donnée des équations considérées peut s'approximer par des combinaisons linéaires à coéfficients constants de ces solutions fondamentales. La détermination de la solution satisfaisant à des conditions initiales données (premier problème au contour pour l'équation elliptique, valeurs de la fonction et de ses dérivées premières sur une courbe coupée une seule fois par les caractéristiques pour le cas hyperbolique) s'effectue alors en considérant les combinaisons linéaires des premières n fonctions fondamentales dont les valeurs au contour satisfont avec approximation croissante aux conditions données. Pour le cas elliptique il suffit pour celà de rendre minimum le maximum de la différence entre les valeurs données et les valeurs que prennent ces combinaisons linéaires, ou bien, plus simplement, l'intégrale d'une puissance convenable du module de cette différence. Dans le cas hyperbolique on considère l'intégrale le long de la courbe donnée de la somme des carrés des différences entre les valeurs de la fonctions et de ses dérivées partielles et les valeurs correspondantes de la fonction d'approximation. Dans le cas de l'équation elliptique, on suppose, en premier lieu, connu à priori que l'équation donnée ne possède une solution $\neq 0$ prenant la valeur 0 au contour; on traite ensuite la détermination des valeurs caractéristiques pour une équation de la forme $L_\lambda(U)=L(U)+\lambda EU=0$ (E fonction donnée positive) en les considérant comme limites de valeurs que rendent minimum la valeur absolue sur le contour d'une solution approximative soumise à une condition de la forme $\iint EU_n^2=1$. Sous l'hipothèse que l'équation soit autoadjointe et que à la valeur caractéristique considérée corresponde une seule solution fondamentale, cette solution est la limite de cette solution approximative. *B. Levi.*

Vecoua, I. Randwertaufgaben in der Theorie der linearen elliptischen Differentialgleichungen mit zwei unabhängigen Veränderlichen. I. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] 1, 29–34 (1940). (Russian) [MF 2016]

The author indicates a new method for the reduction of the calculation of a function U satisfying $\mathbf{L}(U)\equiv U_{z\bar{z}}+2\Re[AU_z]+BU+F=0$, $z=x+iy$, $\bar{z}=x-iy$, $U_z=\partial U/\partial x-i\partial U/\partial y$, etc., in a domain $\mathfrak{F}^2$ and assuming the given values $g(z)$ on the boundary $\mathfrak{f}^1$ of $\mathfrak{F}^2$ to the solution of the integral equation

$$(1)\quad \Re[\alpha(z)k(z)]+(i\pi)^{-1}\Im[\alpha(z)]\int_{\mathfrak{f}^1}k(z_1)(z_1-z)^{-1}dz_1+\int_{\mathfrak{f}^1}P(z,z_1)k(z_1)dz_1=g^*(z),\quad z\in\mathfrak{f}^1.$$

α, β and P are functions depending only on $\mathbf{L}$. The result is based on the representation

$$U(z)=\Re\left[\alpha(z)f(z)+\int_{\mathfrak{F}^2}\beta(z,z_1)f(z_1)dz_1\right],$$

$$f(z)=(i\pi)^{-1}\int_{\mathfrak{f}^1}k(z_1)(z_1-z)^{-1}dz_1$$

being an appropriate analytic function of one complex variable, regular in $\mathfrak{F}^2$. Notwithstanding some gaps in the proof, the result appears to be an interesting application of the connection between solutions of elliptic linear differential equations and complex functions; such a connection has also been pointed out by the reviewer [cf. Math. Z. 32, 386–406 (1930); Rec. Math. [Mat. Sbornik] N.S. 2 (44), 1170–1198 (1937); cf. the preceding review].

S. Bergmann (Cambridge, Mass.).

Vecoua, I. Randwertaufgaben in der Theorie der linearen elliptischen Differentialgleichungen mit zwei unabhängigen Veränderlichen. II. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] 1, 181–186 (1940). (Russian. German summary) [MF 2023]

The author shows that the singular integral equation (1) which appears in the first part of his paper reviewed above is equivalent to the integral equation

$$k(z)-\int_{\mathfrak{f}^1}K(z,\tau)k(\tau)d\sigma(\tau)=g^*(z),\qquad z\in\mathfrak{f}^1,$$

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which is of Fredholm type, where K depends only on $\mathbf{L}$ and $\mathfrak{F}^2$ (in a way indicated by the author); g^* also depends on the boundary values. Thus, using classical results for homogeneous and nonhomogeneous integral equations, the author obtains analogous results for solutions of the first boundary problem of linear differential equations of elliptic type.

S. Bergmann (Cambridge, Mass.).

Weyl, Hermann. The method of orthogonal projection in potential theory. Duke Math. J. **7**, 411–444 (1940). [MF 3411]

If G is a region in three space, and φ a function defined therein, the Dirichlet problem in potential theory is equivalent to the problem of expressing φ in the form $\varphi=\psi+\eta$, where $\psi=0$ on the boundary of G and η is harmonic, for then η must assume the same boundary values as φ. Furthermore ψ and η are orthogonal according to the metric $\int(\text{grad }\varphi)^2$; that is, $\int \text{grad }\psi\cdot\text{grad }\eta=0$, where $\int$ indicates triple integration over G. The author considers the Hilbert space of vector fields f, with metric $\int f^2$. In order to characterize a vector field f as a gradient of a scalar the following set of notions is employed: A scalar field ψ is said to be of class Γ if ψ vanishes in a strip around the boundary of G and has continuous first derivatives. A vector field f is of class Γ if its components are; f is called irrotational if $\int f\cdot\text{rot }v=0$ for every v of Γ, and f is called solenoidal if $\int f\cdot\text{grad }\psi=0$ for every ψ of Γ. Since $\int\text{grad }\psi\cdot\text{rot }v=0$ if ψ and v are of Γ, it is seen that an irrotational f has the characteristics of a gradient, and a solenoidal f those of a rotation. The principal results of this paper may be summarized in the following list: (1) If f is irrotational and solenoidal, f has harmonic components, and rot $f=$div $f=0$. (2) If f is irrotational, then $f=g+e$, where e is irrotational and solenoidal, $g=\lim_{\nu\to\infty}\text{grad }\psi_\nu$, where ψ_ν is in Γ, and $\int e\cdot g=0$. (3) If f is solenoidal, then $f=g+e$, where e is irrotational and solenoidal, $g=\lim_{\nu\to\infty}\text{rot }v_\nu$, where v_ν is in Γ, and $\int e\cdot g=0$. (4) If f is solenoidal and φ is a scalar, then $f=g+e$ and $\varphi=\psi+\eta$, where (a) η and the components of e are harmonic and $e=\text{rot }v$, $\eta=\text{div }v$, (b) $g=\lim_{\nu\to\infty}\text{rot }v_\nu$, $\psi=\lim_{\nu\to\infty}\text{div }v_\nu$, v_ν in Γ, and (c) $\int(e\cdot g+\psi\eta)=0$.

The first two statements contain the boundary value problem of potential theory. If φ is a given function and f its gradient, then f can be written as $f=g+e$, where $e=\text{grad }\eta$, η being harmonic. But $g=\lim_{\nu\to\infty}\text{grad }\psi_\nu$ and the ψ_ν can be shown to converge in the mean to ψ such that $\int_T\psi\,\text{div }p=-\int_T g\cdot p$, where T is any cube and p is any vector with a continuous derivative which has a normal component vanishing on the boundary of T. This condition is the integral equivalent of the fact that $g=\text{grad }\psi$ and $\psi=0$ on the boundary of G. Thus $\varphi=\psi+\eta$ as required. The decompositions (3) and (4) give solutions to other boundary value problems. The method of investigation rests largely on the construction of the proper functions of class Γ to prove statement 1 from the definitions, and the development and use of certain integral inequalities on φ and grad φ. The author also makes considerable investigation into the periods of f, g and e in case they are not single valued.

J. W. Green (Rochester, N. Y.).

Landkoff, N. Sur la structure de l'ensemble des points irréguliers dans le problème de Dirichlet. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 291–293 (1940). [MF 3327]

If Ω is an open set on three dimensions, F its frontier, assumed to be bounded, $f(s)$ a bounded function measurable Borel on F, and $\mu(M, e)$ the distribution of mass on F obtained by sweeping out a unit mass from M, then the generalized solution of the Dirichlet problem for Ω with the boundary values $f(s)$ is given by $\varphi(M)=\int_F f(s)d\mu(M, e_s)$. For a point P of F, $\mu(P, e)$ can be defined by weak convergence: if P is regular, $\mu(P, e)$ is a unit mass at P. If P is an irregular point where $f(s)$ is continuous, the author calls the Dirichlet problem for the boundary values $f(s)$ soluble at P if $\varphi(M)$ is continuous at P. He thus gets as a necessary and sufficient condition for solubility: $f(P)=\int_F f(s)d\mu(P, e_s)$.

Let $\{P_n\}$ be a sequence of irregular points tending to an irregular point P and such that $\{\mu(P_n, e)\}$ tends weakly to a function $\nu(P, e)$. The author proves the theorem that, in order that the solubility of the Dirichlet problem at all the points of $\{P_n\}$ imply the solubility at P, it is necessary and sufficient that the sequence $\{P_n\}$ be not singular; that is, that the function $\nu(P, e)$ does not consist of a unit mass at P. Using this result, the author shows that, from the set F_1 of all irregular points, a denumerable set D can be extracted such that the solubility of the Dirichlet problem at all the points of D implies the solubility at all the points of F_1. The set D contains all the isolated points of F_1, and also all points of another sort, called quasi-isolated points. These are the irregular points characterized by the fact that every sequence of irregular points converging to them is singular. They are of course denumerable.

J. W. Green.

Rapoport, I. Sur le problème plan inverse de la théorie du potentiel. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 305–307 (1940). [MF 3331]

Let r, φ be polar coordinates in a plane, and $\alpha_0, \alpha_2, \cdots, \alpha_n$, $\beta_2, \beta_3, \cdots, \beta_n$ given constants. The paper deals with the problem of finding a domain G so that the function

$$\alpha_0 \log r+\sum_{m=2}^{n}\alpha_m\cos m\varphi+\beta_m\sin m\varphi$$

is the logarithmic potential of a mass of constant density which fills G. The author derives equations for the coefficients of that polynomial

$$f(z)=a_1z+a_2z+\cdots+a_{n+1}z^{n+1},\quad z=re^{i\varphi};\ a_1>0,$$

which maps the unit circle on G. Finally he illustrates his method by giving an example.

E. Rothe.

Dinghas, Alexander. Über positive harmonische Funktionen in einem Halbraum. Math. Z. **46**, 559–570 (1940). [MF 2782]

Generalizing a well-known result on positive harmonic functions in a half-plane, the author proves that any function $u(P)\equiv u(x_1, x_2, \cdots, x_n)$ of n variables $x_1, x_2, \cdots, x_n$ which is harmonic in the hyperplane $x_1>0$ can be represented by a Stieltjes-Radon integral:

$$(1)\qquad u(P)\equiv\frac{2x_1}{\omega_n}\int_{(x_1=0)}\frac{d\Phi(E_Q)}{r}+cx_1,\qquad r=\overline{PQ},$$

where ω_n is the surface area of the hypersphere $\sum_{i=1}^{n}x_i^2=1$, c is a constant and $\Phi\equiv\Phi(E)$ is an absolutely additive set function defined in the hyperplane $x_1=0$. This result is obtained by generalizing to n dimensions the proof given by Tsuji for the case $n=2$ [Jap. J. Math. **15**, 13–18 (1938)]: $u(P)$ is first represented by the Poisson integral for a hypersphere of radius R with center on the x_1-axis and passing through the origin O, and (1) is then derived by a suitable limit process in which $R\to\infty$. In the case

$n=2$ ($u(P)\equiv u(re^{i\varphi})$, $-\pi/2<\varphi<\pi/2$, $r>0$) the author has previously shown [S. B. Preuss. Akad. Wiss. **1935**, 576–596] that

$$m(r)/r=(1/r)\int_{-\pi/2}^{\pi/2} u(re^{i\varphi})\cos\varphi d\varphi \downarrow c \qquad \text{as } r\uparrow\infty$$

(c is the constant in (1) when $n=2$), and Tsuji has added that $m(r)/r$ is a concave function of $1/r^2$. The author proves here the analogous results for $n>2$. Finally, an analogue of the well-known theorem of Wolff-Carathéodory-Landau-Valiron is proved:

$$\lim_{x_1\to\infty} u(P)/x_1=c, \tag{2}$$

where P is in any fixed "angle" $0\leqq\varphi_1\leqq(\pi/2)-\delta$ ($\delta>0$), φ_1 being the angle between $\overline{OP}$ and the x_1-axis. The paper contains a few minor errors and a number of misprints all of which however are easily corrected. [Using the Poisson-Stieltjes representation of a positive harmonic function in a sphere as given by Evans and Bray [Amer. J. Math. **49**, 153–180 (1927)], the reviewer proved a theorem similar to (2) for the case of a 3-dimensional sphere [Math. Ann. **107**, 1–27 (1932)].] *S. E. Warschawski* (St. Louis, Mo.).

Nicolesco, Miron. Sur quelques familles de fonctions polyharmoniques. Acad. Roum. Bull. Sect. Sci. **22**, 478–483 (1940). [MF 3205]

The author extends some of his previous results on polyharmonic functions (solutions of the iterated Laplace equation $\Delta^p u=0$). A family $\{u\}$ of such functions is defined to be completely normal if the families $\{\Delta^r u\}$ are normal for $r=0$ to $p-1$. Likewise $\{u\}$ is said to be completely subharmonic if $(-\Delta)^r u\leqq 0$, for $r=0$ to $p-1$. Any point at which $\{u\}$ fails to be completely normal is called an irregular point. The theorems proved include the following. I. If a family of polyharmonic functions is uniformly bounded in a domain D, or is completely subharmonic in D, or is (locally) completely normal at each interior point of D, then it is completely normal in the interior of D. II. If a family of polyharmonic functions is completely bounded at each point of a domain D, then the set of irregular points of the family consists of a finite number of continua, none of which is separated from the boundary of D. III. The limit of a convergent series of polyharmonic functions is polyharmonic of the same order except at a set of points nowhere dense in the domain of convergence. *O. Frink.*

Spain, B. A boundary problem. Quart. J. Math., Oxford Ser. **11**, 271–272 (1940). [MF 3788]

The following theorem is proved: If D is a closed region of the plane bounded by a curve C consisting of a finite number of arcs with continuously turning tangent, and if $f(x, y)$ is a function defined in D such that (i) $f=0$ on C, (ii) grad. $f=0$ on C, (iii) $\nabla^2 f\geqq 0$ on C, the equality sign holding at most at a finite number of points, (iv) $\nabla^2(\nabla^2 f)\geqq 0$ in D, (v) f and all its derivatives up to and including those of the third order are continuous within and on the boundary of D, then $f\geqq 0$ in D, the equality sign not holding in any subdomain where $\nabla^2(\nabla^2 f)>0$. The author notes that the theorem can be extended for any number of dimensions. In the eighth line on page 272, the word "maximum" should be replaced by "minimum." *F. W. Perkins.*

Piskounov, N. Intégration des équations de la théorie des couches frontières. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 104–106 (1940). [MF 3211]

The present note deals with the equation of the boundary layer in the reduced form given by von Mises [Z. Angew. Math. Mech. **7**, 425–431 (1927)], namely with $\mathbf{L}(U)\equiv U_{yy}-(a-U)^{-1}U_x=0$, and the boundary conditions $U(0, y)=\phi(y)$, $U(x, 0)=a$, $\lim_{y\to\infty} U(x, y)=0$. The author announces the following results: (I) If U satisfies $\mathbf{L}(U)=0$ in $A=\mathrm{E}[x\leqq X, -R\leqq y\leqq R]$, then there exist upper and lower bounds for the derivatives of U depending only on the maximum of U in A. (II) If ϕ satisfies the hypotheses: $\phi(0)=1$, $\lim_{y\to\infty}\phi(y)=0$, $\phi(y)<1$ for $y>0$, $\phi(y)<-ky+1$ for $y<k_1$ ($k>0$, $k_1>0$, arbitrarily small), then there exists a solution of $\mathbf{L}(U)=0$ satisfying the above boundary conditions. (III) If in addition to the hypotheses of (II), $\phi''(y)\geqq 0$ for $y\leqq\epsilon$, ϵ arbitrary, then the solution is unique. The reviewer observes that Luckert gave a method for a numerical solution of $\mathbf{L}(U)=0$ [Schr. Math. Inst. u. Inst. Angew. Math. Univ. Berlin **1**, 245–274 (1933)]. *S. Bergmann.*

Piscounov, N. Problèmes limites pour les équations du type elliptico-parabolique. Rec. Math. [Mat. Sbornik] N.S. **7** (**49**), 385–424 (1940). (Russian. French summary) [MF 2800]

The author considers the equation

$$\mathbf{L}(u)\equiv u_{xx}-\sum_{i=1}^{m} a_i u_{y_i}-cu_x-bu+f=0,$$

where a_i, c, b, f are functions of x, y_1, $\cdots$, y_m. Let $\mathfrak{B}$ be a domain in the x, y_1, $\cdots$, y_m-space bounded by a hypersurface $\mathfrak{S}$ which is homeomorphic to a sphere and such that the parallel to the x-axis through any interior point of $\mathfrak{B}$ cuts $\mathfrak{S}$ in two points. Denote the totality of such intersection points by $\mathfrak{S}_1''$, and write $\mathfrak{S}'=\mathfrak{S}-\mathfrak{S}_1''$. Let $\mathfrak{S}_1'$ be the part of $\mathfrak{S}'$ through which lines $\mathrm{E}[dy_1/a_1=dy_2/a_2=\cdots=dy_m/a_m]$, the bi-characteristics of $\mathbf{L}$, enter $\mathfrak{B}$. The author proves under appropriate hypotheses that there exists one and only one solution of $\mathbf{L}(u)=0$ in $\mathfrak{B}$ assuming on $\mathfrak{S}_1''+\mathfrak{S}_1'$ given values u_0. He distinguishes two cases: (a) $\mathfrak{S}_1'$ is a $(m-1)$-dimensional continuum which does not contain intervals parallel to the x-axis, (b) $\mathfrak{S}_1'$ is a cylindric m-dimensional surface. In the case (b) it is possible to reduce the problem by certain transformations to the case where $\mathfrak{S}_1''$ consists of two domains lying in $x=0$ and $x=1$, respectively, and $\mathfrak{S}_1'$ lies in $y_1=0$; furthermore, it is possible to suppose that the given values u_0 vanish on $\mathfrak{S}_1''$ and that $c=0$. The following result is then valid. Let the given function $u_0(x, y_1, \cdots, y_m)$ possess derivatives of the fourth order; let $u_0(0, 0, y_2, \cdots, y_m)=u_0(1, 0, y_2, \cdots, y_m)=0$; let $c=0$ and a_i and f possess continuous derivatives of the third order. Then $\mathbf{L}(u)=0$ admits one and only one solution in $\mathfrak{B}$ assuming on $\mathfrak{S}_1''+\mathfrak{S}_1'$ the given values u_0. The proof is roughly as follows. First of all $\mathbf{L}(u)$ can be reduced to the form

$$\mathbf{L}_1(u)\equiv u_{xx}-(\textstyle\sum a_i^2)^{\frac{1}{2}}u_N-bu-f=0,$$

where $u_N=\partial u/\partial N$ denotes the derivative in the direction defined by the bi-characteristics. Then $\mathfrak{B}$ is divided by the parallel planes $y_1=\nu h$, $\nu=1, 2, \cdots, n$, and $\mathbf{L}_1(u)=0$ is replaced by a sequence of n ordinary differential equations

$$\partial^2 u_{\nu+1}^{(h)}/\partial x^2-a_{\nu+1}(u_{\nu+1}^{(h)}-u_\nu^*)/h-b_{\nu+1}u_\nu^*+f_{\nu+1}=0.$$

Here u_ν^* are functions which can be expressed in terms of $u_\nu^{(h)}$. The author then proves that each equation has a solution and that, for $h\to 0$, $u_\nu^{(h)}$ converge to a function u satisfying $\mathbf{L}_1(u)=0$ and the boundary conditions. The proof is related to the method of Eric Rothe (which in turn arose from considerations of von Mises) for treating the equation $u_{xx}-au_y-f=0$ [Math. Ann. **102**, 650–670 (1930)].

The author also treats other cases, as for example the case (a), and obtains further existence theorems.

S. Bergmann (Cambridge, Mass.).

Karimov, Dsh. H. Über die periodischen Lösungen der nichtlinearen differentialen Gleichungen des parabolischen Typus. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 403–406 (1940). [MF 3609]

The author treats the following periodic boundary value problem

$$\frac{\partial}{\partial x}\left[P(x)\frac{\partial Z}{\partial x}\right]-\frac{\partial Z}{\partial t}=\sum_{n=0}^{\infty}\Phi_n(t)X_n(x)+\mu f(Z),$$

$$Z(0,t)=Z(\pi,t)=0;\quad Z(x,0)=Z(x,1).$$

$\Phi_n(t)=\Phi_n(t+1)$ and X_n is a normalized characteristic function of the Sturm-Liouville system

$$\frac{d}{dx}\left[P\frac{dX}{dx}\right]+\lambda X=0,\qquad X(0)=X(\pi)=0.$$

The case $P(x)=1$ was treated by Dj. Kh. Karimov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **25**, 3–6 (1939); cf. these Rev. **1**, 315]. *F. G. Dressel* (Durham, N. C.).

Dressel, F. G. The fundamental solution of the parabolic equation. Duke Math. J. **7**, 186–203 (1940). [MF 3397]

So-called fundamental solutions (playing the same rôle as $(y-\eta)^{-\frac{1}{2}}\exp(-(x-\xi)^2/4(y-\eta))$ in the theory of the one dimensional heat equation) for linear parabolic equations have been constructed by J. Hadamard, M. Gevrey, W. Feller and the present reviewer. The author constructs a fundamental solution for the following linear parabolic equation which is more general than those treated by these writers:

$$\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n}a_i\frac{\partial u}{\partial x_i}+au-\frac{\partial u}{\partial y}=0.$$

The coefficients are functions of $x_1, x_2, \cdots, x_n$ and y, and satisfy certain differentiability and Lipschitz conditions; moreover, $\sum a_{ij}\xi_i\xi_j$ is supposed to be a positive definite form. The method employed by the author follows the lines of E. E. Levi's construction of a fundamental solution in the elliptic case [Rend. Circ. Mat. Palermo **24**, 275–317 (1907)]. Application to boundary value problems will be made in a later paper. *E. Rothe* (Oskaloosa, Iowa).

Fjeldstad, J. E. A generalization of Abel's integral equation. Norsk Mat. Tidsskr. **22**, 41–51 (1940). (Norwegian) [MF 3313]

This paper reproduces an address in which the author considers different integral equations connected with the equation for the conduction of heat. Consider for example two solutions which vanish for $t=0$ and for $x\to\infty$, and which satisfy the conditions $u(t,0)=f(t)$ and $u_x(t,0)=-\phi(t)$, respectively. The question of when these solutions are identical leads to an integral equation of Abel's type.

W. Feller (Providence, R. I.).

Churchill, R. V. A heat conduction problem introduced by C. J. Tranter. Philos. Mag. (7) **31**, 81–87 (1941). [MF 3756]

The problem of finding the temperatures in a composite semi-infinite solid, with initial temperature A in $0<x<a$ and zero in the strip $x>a$, is here solved by means of the Laplace transformation. *F. G. Dressel* (Durham, N. C.).

Jaeger, J. C. Radial heat flow in circular cylinders with a general boundary condition. J. Proc. Roy. Soc. New South Wales **74**, 342–352 (1940). [MF 3780]

Formulas are derived for the temperatures $v(r,t)$ in each of the three types of cylindrical solids $r\leqq a$, $a\leqq r\leqq b$ and $r\geqq b$ when the temperature at the boundary surface satisfies the condition

$$k_1\frac{\partial v}{\partial t}+k_2\frac{\partial v}{\partial r}+k_3v=k_4.$$

Both the case in which the initial temperature is zero and that of an instantaneous source at $t=0$ on any cylindrical surface $r=r'$ are treated. The result in the latter case leads to the temperature formula for the case $v(r,0)=f(r)$, and thus to formal statements of generalized expansion formulas for $f(r)$ in series and integrals of Bessel functions. The results are obtained by using the Laplace transformation. A brief discussion is given of the practical importance of the above type of temperature condition at the surface. Such conditions may arise, for instance, when the cylinder is in contact with a fluid, as in hot water systems. *R. V. Churchill.*

Carslaw, H. S. A simple application of the Laplace transformation. Philos. Mag. (7) **30**, 414–417 (1940). [MF 3692]

A semi-infinite solid at uniform temperature is placed in contact with another such solid, or with a slab of finite thickness, of another material and with a different uniform temperature. By means of the Laplace transformation, formulas are derived for the temperatures in the two composite solids so formed. These are two problems that were treated recently by Tranter [Philos. Mag. (7) **28**, 579–583 (1939); these Rev. **1**, 181], using contour integrals; but the method used in the present note has obvious advantages over the method of contour integration. *R. V. Churchill.*

Thiruvenkatachar, V. R. Operational solution of the equation of telegraphy. J. Mysore Univ. Sect. B. **1**, 9–13 (1940). [MF 3623]

The solution of the differential equation

$$\nabla^2 u=u_{tt}+u \tag{1}$$

taking on the initial conditions $u=0$, $u_t=\varphi$ for $t=0$ is obtained in the form

$$u_P=\frac{e^{t/2}}{t}\frac{\partial}{\partial t}\int_0^t\rho^2\bar{\varphi}(\rho)J_0\left(\frac{i}{2}\sqrt{t^2-\rho^2}\right)d\rho. \tag{2}$$

Here $\bar{\varphi}(\rho)$ is the mean value of φ over a unit sphere of radius ρ and center at P. The method used consists in taking the means of u about P over spheres with center at P, then utilizing the Laplace transform of the mean.

H. Poritsky (Schenectady, N. Y.).

John, Fritz. The Dirichlet problem for a hyperbolic equation. Amer. J. Math. **63**, 141–154 (1941). [MF 3638]

The writer considers the Dirichlet problem for (1) $u_{xy}=0$ (or rather the twice integrated form to avoid differentiability adversions) with data assigned on a Jordan curve C convex in the x and y directions. The analysis combines the classic device of a chain of characteristics abutting on C with the modern theory of topological transformations of C into itself. From $P\varepsilon C$ proceed along a horizontal characteristic to the next intersection point with C and then along the vertical characteristic to $P'\varepsilon C$. Write $P'=TP$. If T is a 1-1, even and continuous transformation of C into itself,

four cases are possible. C is "periodic" if $T^n x = x$ for some n and all x and is "transitive" if no x is periodic and $\{T^n x\}$, $n=0, 1, \cdots$, is dense on C. The theory for the rectangle essentially includes these two cases only. There is a homeomorphism of transitive C and its interior, preserving (1), onto the rectangle of side slope ± 1 and length breadth ratio ξ, an irrational number. The discussion of Bourgin-Duffin [Bull. Amer. Math. Soc. **45**, 851–858 (1939); these Rev. **1**, 120] thus applies. Moreover "some additions to the observations in that paper" are made. The solution of the Dirichlet problem is unique unless C contains a non-denumerable set of periodic points. This generalizes a condition for the rectangle to non-periodic, non-transitive C. Overdetermination for non-transitive C occurs in the sense that a proper sub-arc of C exists such that data on C consistent with the solution are uniquely determined when given merely for the sub-arc. *D. G. Bourgin.*

Mindline, J. A. La diffraction d'une onde plane par rapport à un cercle. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 556–560 (1940). [MF 3566]

For the representation of the perturbation in the region outside the circle use is made of a series of type

$$u=\sum_{n=0}^{\infty}\int_{-\infty}^{at-r}[A_n(z)\cos n\theta+B_n(z)\sin n\theta]T_n(\operatorname{ch}\omega)dz/(r\operatorname{sh}\omega),$$

where $T_n(\operatorname{ch}\omega)=\operatorname{ch} n\omega$ is Tchebycheff's polynomial and at $-z$ is equal to $r\operatorname{ch}\omega$. The boundary conditions give rise to integral equations of type

$$f(x)=\int_{-1}^{s}k(x-z)\phi(z)dz$$

for the determination of $A_n(z)$ and $B_n(z)$, where

$$[(s+1)^2-1]^{\frac{1}{2}}k(s)=T_n(s+1).$$

These equations are inverted with the aid of a function $H(x-z)$. Similar analysis is developed for an elastic problem. *H. Bateman* (Pasadena, Calif.).

Mindline, J. A. Sur la distribution des ondes dans l'espace à trois dimensions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 561–565 (1940). [MF 3567]

The wave-equation $U_{xx}+U_{yy}+U_{zz}=U_{tt}$ has a solution which is represented in polar coordinates by the integral

$$U=\int_0^{\infty}[A_n^{(1)}(t-r\operatorname{ch}\xi)+A_n^{(2)}(t+r\operatorname{ch}\xi)]\times P_n(\operatorname{ch}\xi)\operatorname{sh}\xi d\xi\cdot Y_n(\theta,\phi),$$

where $A_n^{(1)}(s)$, $A_n^{(2)}(s)$ are suitable arbitrary functions. For the representation of a diverging wave an appropriate choice is $A_n^{(2)}(s)=0$. A series is used to represent a solution which vanishes at infinity. The unknown functions $\Phi_{n,m}^{(i)}(r\operatorname{ch}\xi, t)$ occurring as coefficients in the series are determined by the integral equations

$$a_{n,m}^{(i)}(r,t)=\int_0^{\infty}\Phi_{n,m}^{(i)}(r\operatorname{ch}\xi,t)P_n(\operatorname{ch}\xi)\operatorname{sh}\xi d\xi,$$

$$i=1,2;\ n=0,1,2,\cdots;\ m=0,1,\cdots,n.$$

H. Bateman (Pasadena, Calif.).

Mindline, J. A. La résolution du problème extérieur de Cauchy-Dirichlet pour une équation ondulatoire dans le cas d'une sphère. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 566–569 (1940). [MF 3568]

The aim is to find a solution of $U_{xx}+U_{yy}+U_{zz}=U_{tt}$ which satisfies the initial conditions $U=U_0(r,\theta,\phi)$, $U_t=U_0^{(1)}(r,\theta,\phi)$ when $t=0$ and the boundary condition $U=F(\theta,\phi,t)$ when $r=1$, U_0, U_0' and F being represented by series of spherical harmonics. Use is made of a type of integral considered in the paper reviewed above and the solution is effected with the aid of integral equations. *H. Bateman.*

Mindline, J. A. La diffraction d'une onde plane par rapport à une sphère. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 940–945 (1940). [MF 3588]

The wave-function U of the incident waves and the wave-function u of the perturbation produced by the sphere are such that $U+u=0$ for $r=1$ when the sphere $r=1$ is a fixed boundary, and $U_r+u_r=0$ for $r=1$ when the boundary is free. If

$$U=f(at+x\sin\lambda\cos v+y\sin\lambda\sin v+z\cos\lambda)$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}[a_{n,m}(r,t)\cos m\phi+b_{n,m}(r,t)\sin m\phi]P_n^{(m)}(\cos\theta),$$

an appropriate expression for u is a similar summation in which the quantity within square brackets is

$$\int_{-\infty}^{at-r}\{A_{n,m}(z)\cos m\phi+B_{n,m}(z)\sin m\phi\}P_n\left(\frac{at-z}{r}\right)dz/r,$$

and $A_{n,m}(z)$, $B_{n,m}(z)$ are determined by integral equations of Volterra's type in which the kernel is $P_n'(x-z+1)$. These equations are solved with the aid of a function $L_{n,m}(x-s)$ and the corresponding analysis is given for the diffraction of an elastic wave. *H. Bateman* (Pasadena, Calif.).

MECHANICS

Dynamics, Celestial Mechanics

Sauer, Robert. Graphische Statik räumlicher Kräftesysteme mit Hilfe der dualen Kräfteabbildung. Z. Angew. Math. Mech. **20**, 174–180 (1940).

Sestini, Giorgio. Composizioni di moti rigidi. Boll. Un. Mat. Ital. (2) **2**, 235–240 (1940). [MF 2980]

Arrighi, Gino. Sulle reazioni vincolari d'impulso. Boll. Un. Mat. Ital. (2) **2**, 233–235 (1940). [MF 2979]

Gorshkov, P. M. Differential equation of the motion of a pendulum on a moving support. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **9**, 172–180 (1940). (Russian) [MF 3298]

Lurie, A. I. Problem of forced non-linear oscillations. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **8**, 25–33 (1939). (Russian) [MF 3339]

The author considers some cases of forced non-linear vibrations of systems characterized by the equation

$$\ddot{x}+f(x,\dot{x})=\varphi(t),$$

where the force function $\varphi(t)$ is a periodic function of the time t of period $T=2\pi/\omega$. The steady-state solutions of period T can be determined from a certain integral equation, the approximate solution of which is given in the form of a trigonometric polynomial. *I. S. Sokolnikoff.*

Biezeno, C. B. Critical speeds of rotating shafts. Nederl. Akad. Wetensch., Proc. **43**, 1144–1152 (1940). [MF 3724]

The known theorem that a massless rotating shaft, bear-

ing n centered and concentrated masses, has at most n critical whirling speeds is refined by proving that the critical speeds are all different, provided all supports are rigid and all but the end supports transmit no moments to the shaft. (It is shown by simple examples that these restrictive conditions are necessary.) The proof of the fact that the secular equation has no multiple roots is accomplished by showing that there are no common roots of the secular equation and an equation obtained by equating to zero a certain $(n-1)$st order minor of the secular determinant. *E. Reissner.*

Magnus, K. Untersuchungen zur Verminderung störender Rüttelschwingungen an Kreiselgeräten. Z. Angew. Math. Mech. **20**, 165–174 (1940).

The author determines the choice of the moments of inertia of a top-like figure, so that the vibrations induced by the centrifugal moments remain as small as possible. In order that there be no resonance phenomenon between the rotation of the top and the nutation frequency, certain relations between the moments of inertia of the top and the suspension must be fulfilled. This condition tells us that the moment of inertia of the suspension must have a determined minimum value. Special examples are considered.

A. E. Heins (West Lafayette, Ind.).

***Roever, William H. The Weight Field of Force of the Earth.** Washington University Studies, New Series, Science and Technology, no. 11, 1940. iv+84 pp. $1.50.

One of the classical problems in mechanics deals with the effect of the rotation of the earth upon the motion of a body [cf. Appell, "Traité de Mécanique Rationnelle," vol. II, chapter 22, especially pages 272–285, 1904 edition]. This monograph by Roever discusses such problems. In particular the following general topics are discussed: (a) relative motion, (b) gravitational effects, (c) geodetic considerations and (d) the motion of a projectile. Special examples are carried out in detail and a numerical example is considered.

A. E. Heins (West Lafayette, Ind.).

Spain, B. Note on the hodograph of non-holonomic dynamical systems. Proc. Edinburgh Math. Soc. (2) **6**, 176–180 (1940). [MF 2852]

The author attempts to generalize for non-holonomic systems the theory of the hodograph for holonomic systems given by Synge [Trans. Roy. Soc. Canada, Sect. 3. **25**, 121 (1931)]. The generalization proves complicated, except in the case where the constraint-vectors satisfy certain conditions; in that case the equations of motion take the holonomic form with a modified force-vector and the former theory can be carried over completely. The conditions on the constraint-vectors imply that the acceleration (as well as the velocity) is perpendicular to the local subspace defined by the vectors of constraint. *J. L. Synge.*

Nagabhushanam, K. On a property of the Lagrangian. J. Indian Math. Soc. (N.S.) **4**, 89–90 (1940). [MF 3270]

The author, following ideas expressed in earlier papers [J. Indian Math. Soc. **20**, 236–243 (1934) and Proc. Indian Acad. Sci., Sect. A. **1**, 555–561 (1935)], considers the trajectories of a conservative holonomic dynamical system of n degrees of freedom in a manifold of states A_{2n+1} of coordinates (p_r, q_r, t), $r=1, 2, \cdots, n$. The action form $\sum p_i dq^i - Hdt$ takes, in general coordinates, the form $X_i dx^i$. In this paper two special forms of the integral invariant

$$\int \cdots \int (X_i \alpha^i) dx^1 dx^2 \cdots dx^{2n+1}$$

are obtained, where α^i is a contravariant vector density codirectional with the trajectories. *D. J. Struik.*

Williamson, John. An algebraic problem involving the involutory integrals of linear dynamical systems. Amer. J. Math. **62**, 881–911 (1940). [MF 2888]

It is well known that any Hamiltonian system of n degrees of freedom admits in the small exactly n independent integrals in involution, including the integral of energy. It is the main purpose of this paper to prove a conjecture of Wintner to the effect that, in case the system is a linear system (that is, the Hamiltonian function is a quadratic form), it is always possible to find n independent quadratic forms, which are n integrals in involution. Although, in the more or less general case of simple elementary divisors, the theorem is almost obvious, any complete proof would appear to be necessarily rather involved. The present proof is purely algebraic and is based, to a large extent, on results, previously proved by the author, which give normal forms for a pencil of matrices, whose base consists of a symmetric and a non-singular skew symmetric matrix [cf. Amer. J. Math. **58**, 141–163 (1936)]. The existence of further quadratic integrals is also investigated and an application is made to the equations of variation from an equilateral Lagrangian libration point in the restricted problem of three bodies. *D. C. Lewis* (Durham, N. H.).

Madhava Rao, B. S. On an invariant relation of dynamical systems. J. Mysore Univ. Sect. B. **1**, 1–3 (1940). [MF 3621]

There are given here two slightly simplified proofs of the Wintner-Pars identity,

$$\frac{d}{dt}(\sum g_{ij}q_i\dot{q}_j) = (\beta-\alpha+2)U-(\alpha-2)h,$$

holding for Hamiltonian systems defined by

$$H=\tfrac{1}{2}\sum g^{ij}p_ip_j-U,$$

where the g^{ij} and U are homogeneous of degree α and β, respectively, in the q's. *D. C. Lewis* (Durham, N. H.).

Blackall, Clair J. On volume integral invariants of non-holonomic dynamical systems. Amer. J. Math. **63**, 155–168 (1941). [MF 3639]

All the simple examples of conservative analytic non-holonomic systems with a physical background admit analytic volume integral invariants. In this paper it is shown by means of an example that this is not implied by the mathematical definition of such systems, at least in so far as concerns the existence of a volume integral invariant analytic in the neighborhood of an equilibrium point. The system afforded by the rolling (without slipping) of a uniform sphere on a spherical surface under an arbitrary potential function, which from this point of view is a fairly complicated physical system, is also studied in some detail. The system is shown to possess an analytic integral invariant valid on the constraint manifold. This result can, without additional computation, be extended to the case of an arbitrary system of forces that are functions of the coordinates only, the existence of the potential function being superfluous. The question as to whether or not there are certain characteristics of physical systems which imply the existence of a volume integral invariant remains open.

D. C. Lewis (Durham, N. H.).

Badalian, G. Sur la simplification de l'équation de la trajectoire dans le problème de deux centres fixes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 114–117 (1939). [MF 2917]

An explicit set of substitutions is given which reduces to normal form the elliptic integrals which occur in the problem of two fixed attracting centers. *P. Franklin.*

Sokoloff, George. Sur le mouvement symétrique d'un système de points matériels qui agissent avec des forces dépendentes des distances mutuelles. Acad. Sci. RSS Ukraine. Rec. Trav. [Zbirnik Prace] Inst. Math. **1940**, no. 5, 99–115 (1940). (Ukrainian. Russian and French summaries) [MF 3752]

The author studies trajectories with collisions for the three body problem, when attraction (or repulsion) is proportional to a function $f(\Delta_{i,j})$ of distances and when $\overline{P_0P_1}=\overline{P_0P_2}$; $f(r)=-(d/dr)F(r)$, $\lim r^{2\alpha+1}f(r)=2\alpha$, $\lim r^{2\alpha}F(r)=1$ (as $r\to+\infty$) and $\alpha>0$. The planar relative motion of P_1 with respect to P_0 is considered; the axis of symmetry of the triangle is taken as the x axis and the case of collision of P_1 and P_2 is studied. On letting m_0 denote the mass at P_0 and M be the sum of the three masses, r, φ, R, Φ are introduced by the relations: $\sqrt{m_0}x=r\cos\varphi$, $\sqrt{M}y=r\sin\varphi$, $R=r^\alpha dr/dt$, $\Phi=r^{\alpha+1}dy/dt$. A study of these functions is given ($\alpha\neq1$) in the vicinity of the value $t=t_1$, corresponding to the general collision of the three bodies; it is shown that R, φ, Φ tend to finite limits when $t\to t_1$. In the case when $r^{2\alpha}F(r)$, $r^{2\alpha+1}f(r)$ are developable in series of positive integral powers of r^{α_2}, r^{α_3}, $\cdots$, r^{α_n} (α_2, $\cdots$, $\alpha_n>0$) the form of the solutions is given in the neighborhood of $r=0$ ($t=t_1$).

W. J. Trjitzinsky (Princeton, N. J.).

Sokoloff, G. Sur la collision générale des trois corps qui s'attirent mutuellement proportionnellement à l'inverse d'une puissance quelconque de la distance. Acad. Sci. RSS Ukraine. Rec. Trav. [Zbirnik Prace] Inst. Math. **1940**, no. 4, 7–45 (1940). (Ukrainian. Russian and French summaries) [MF 3230]

The author studies trajectories, with general collision, of the three body problem, when attraction is inversely proportional to the $(2\alpha+1)$st ($\alpha>0$) power of distance. Some of the quantities introduced are as follows: $M=m_0+m_1+m_2$, $(m_0+m_1)\mu_2=m_0m_1$,

$$U=m_0m_1r^{-2\alpha}+m_1m_2r_0^{-2\alpha}+m_2m_0r_1^{-2\alpha},$$
$$MI^2=m_0m_1r^2+m_1m_2r_0^2+m_2m_0r_1^2,$$
$$R=I^\alpha dI/dt,\qquad X=I^{\alpha+1}d\chi/dt$$

[$\mu_2^{\frac{1}{2}}r=I\cos\chi$], $F=I^{2\alpha}U$. Here r_0, r_1, r denote the distances P_1P_2, P_2P_0, P_0P_1, respectively. Studying the behavior of the solutions in the neighborhood of the instant of collision, the author establishes that, if there exists a finite instant t_1 so that $I\to0$ as $t\to t_1$, then: (1) I is monotonically decreasing in some interval (t_0, t_1); (2) if $\alpha\neq1$, then R^2 and $2F$ tend to a common positive limit, as $t\to t_1$; (3) X and four other functions $\to0$ ($\alpha\neq1$), when $\alpha<1$ the constants in the integrals of areas are zero and the motion is planar; (4) the limiting configurations, found by Sundman for $\alpha=\frac{1}{2}$, are an equilateral triangle and a straight line configuration (with a certain condition regarding the ratio of distances and masses). The equations of the planar problem are transformed into a first order differential system, with I for independent variable; the form of the solutions, with respect to each of the limiting configurations, is established in the neighborhood of $I=0$. *W. J. Trjitzinsky.*

Nikitin, P. and Tschebotareff, G. Method of Leverrier and its application to the computation of Ceres by Neptune. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **9**, 93–103 (1940). (Russian) [MF 3297]

Fricke, Walter. Der Einfluss eines widerstehenden Mittels in der Dynamik dichter Sternsysteme. Z. Astrophys. **19**, 304–338 (1940). [MF 2363]

In the first part of the paper, it is shown that the accretion of mass by stars in their motion through interstellar matter will give rise to a dissipative force acting on them. With suitable approximations, the dissipative term $W(\mathbf{v})$ in the equation of motion

$$\frac{d\mathbf{v}}{dt}=W(\mathbf{v})-\text{grad } U, \tag{1}$$

where U is the general gravitational potential, is expressed in the form

$$W(\mathbf{v})=-\frac{\mathbf{v}}{A+B|\mathbf{v}|}, \tag{2}$$

where A and B are certain functions of position and time. The Boltzmann integro-differential equation for the distribution function f is then considered in conjunction with the equations of motion (1) and (2). Particular attention is devoted to consider the circumstances under which the Boltzmann equation will admit of a solution of the form e^{-Q}, where Q is a general quadratic form in the velocities. It is shown that solutions of this form exist only when the dissipative forces are directly proportional to the velocities (that is, the case when a Rayleigh dissipation function can be introduced). The explicit solutions for the various coefficients in the quadratic form are obtained and the case of spherical symmetry is considered in some detail.

S. Chandrasekhar (Williams Bay, Wis.).

Relativity

Sulaiman, Shah. Modification of a relativity postulate. Philos. Mag. (7) **30**, 49–54 (1940). [MF 2567]

The Einstein theory of gravitation predicts that light passing the limb of the sun should be deflected through an angle 1″.74. Sulaiman here quotes evidence to show that the observed values range from 2″.16 to 2″.72, and is led to re-examine the postulate that light travels in a minimal geodesic. On assuming that the areal proper-velocity $h/2=cR/d$ (instead of ∞ as in the Einstein theory), where R is the radius of the sun, Sulaiman obtains a predicted deflection 2″.61, half again as large as Einstein's.

H. P. Robertson (Princeton, N. J.).

Mathisson, Myron. The variational equation of relativistic dynamics. Proc. Cambridge Philos. Soc. **36**, 331–350 (1940). [MF 2602]

In the special theory of relativity an energy field is described by a tensor $T^{\alpha\beta}$ satisfying the conservation equations (*) $\partial_\beta T^{\alpha\beta}=0$. This paper deals with the general problem of the "passage to mechanics," in which the partial differential equations (*) are to be transformed into the ordinary differential equations defining the motion of a particle in, and at least in part responsible for, the field. This transition is to be accomplished with the aid of a lemma obtained [§§ 1, 2] by integrating $T^{\alpha\beta}$, multiplied by a suitably chosen

auxiliary tensor $p_{\alpha\beta}$, over a world-tube containing the particle, and reducing this four-fold integral to a line integral which vanishes in virtue of equations (*). It is shown [§§ 3, 4] that in case $T^{\alpha\beta}$ represents only the material field of the particle, the vanishing of the line integral implies the constancy of a suitably defined momentum vector. The method is extended [§§ 5–9] to the case in which $T^{\alpha\beta}$ is not assumed to be localized in the vicinity of the world-line of the particle, when the line integral contains then a "force term." The last section modifies this extension for cases in which the energy tensor becomes infinite on the central world-line. [For an alternative, but related, attack on this general problem, cf. a paper by the reviewer, Proc. Edinburgh Math. Soc. 5, 63–81 (1936).] *H. P. Robertson.*

Randers, Gunnar. On an asymmetrical metric in the four-space of general relativity. Phys. Rev. (2) **59**, 195–199 (1941). [MF 3699]

The author shows that the theory of a four-dimensional space in which the distance is given by $|ds|$, where

$$ds=k_\mu dx^\mu+(g_{\mu\nu}dx^\mu dx^\nu)^{\frac{1}{2}}, \qquad \mu, \nu=1, 2, 3, 4,$$

$g_{\mu\nu}$ is an indefinite symmetric tensor and k_μ a vector, is equivalent to the theory of the five-dimensional space V_5 used in unified field theories, namely, that space based on the metric tensor γ_{mn} ($m, n=1, 2, \cdots, 5$), with $\gamma_{\mu\nu}=g_{\mu\nu}-k_\mu k_\nu$, $\gamma_{\mu 5}=\gamma_{5\mu}=k_\mu$, $\gamma_{55}=-1$. The author next considers some special V_5's whose Einstein-Ricci tensors G_l^m satisfy

$$G_l^m G_m^p=\theta\delta_l^p. \tag{1}$$

That is, the tensor G_l^m is proportional to a (5×5) orthogonal matrix whose components are functions of coordinates. Thus the G_l^m depend on eleven arbitrary functions and no method is given in the paper for determining the G_l^m further. The author selects these functions so that the orthogonal matrix is a simple diagonal one and then calculates $g_{\mu\nu}$ and k_μ. One of the cases treated gives solutions of the type known as Nordström's solution for the field of a charged particle. Other solutions are also obtained. Because of this the author states "The fundamental equation (1) is seen to be a generalization of Einstein's equation for empty space." Since the G_l^m which satisfy (1) are to a great extent arbitrary, it is evident that this "generalization" is unsuitable for any physical theory. *A. H. Taub* (Princeton, N. J.).

Itimaru, Kusuo. Cosmology in terms of wave geometry. VI. Physical Interpretations. (continued) J. Sci. Hirosima Univ. Ser. A. **10**, 151–156 (1940). [MF 3382]

Takeno, Hyôitirô. Cosmology and conformally flat space. J. Sci. Hirosima Univ. Ser. A. **10**, 173–214 (1940). [MF 3384]

Iwatsuki, Toranosuke and Sibata, Takasi. On some characters of time. J. Sci. Hirosima Univ. Ser. A. **10**, 247–252 (1940). [MF 3386]

The basic ideas of the theory of wave geometry as enunciated by the Hirosima school are the following: The metric

$$ds^2=g_{ij}dx^i dx^j \tag{1}$$

of a Riemannian space is called the macroscopic metric and is to be replaced by the microscopic metric:

$$ds\psi=\gamma_i dx^i\psi, \tag{2}$$

where the γ_i are a set of matrices well-known in spinor theory and satisfy

$$\gamma_i\gamma_j+\gamma_j\gamma_i=g_{ij}1, \tag{3}$$

and ψ is a spinor (to be interpreted analogously to the Dirac wave functions) which is a function of coordinates. It is required to satisfy an equation of the form

$$\psi_{,i}=\Sigma_i\psi, \tag{4}$$

where the comma denotes the covariant derivative. The spin connection is determined in the usual way:

$$\gamma_{i,j}=\partial\gamma_i/\partial x^j-\gamma_k\{{k \atop ij}\}-\gamma_i\Gamma_j+\Gamma_j\gamma_i=0. \tag{5}$$

In some of the earlier papers the Christoffel symbols in this equation are replaced by more general affine connections. The integrability conditions of equations (4) restrict the type of Riemannian space under consideration. It can readily be shown that if equations (5) hold and $\Sigma_i=0$ in (4), then the space must be a flat space if (4) is to be completely integrable.

This theory is applied to cosmology by requiring that equations (4) be completely integrable and that Σ_i be chosen so that the tangent vector to a geodesic in the space with g_{ij} as metric tensor is given by $u^i=\psi^+A\gamma^i\psi$, where the $^+$ denotes the hermitian conjugate and A is a matrix such that $(A\gamma^i)^+=A\gamma^i$. Sibata has shown [the same J. 8, 199–204 (1938)] that these conditions imply that equation (4) be of one of two forms:

$$\psi_{,i}=f\gamma_i\psi, \tag{6}$$

where f is a constant; the conditions of complete integrability of these equations are

$$K^{hilm}=4f^2(g^{il}g^{hm}-g^{hl}g^{im}), \tag{7}$$

where K^{hilm} is the Riemann curvature tensor. The other form of (4) is

$$\psi_{,i}=g^{1/2}\epsilon_{ijmk}\varphi^k\gamma^j\gamma^m\psi, \tag{8}$$

whose conditions of complete integrability are

$$K_{ijlm}=16g\epsilon_{ijpu}\epsilon_{lmqv}\varphi^u\varphi^p g^{qv}, \qquad \varphi_{h,i}=0. \tag{9}$$

In the present paper Takeno obtains

$$K_{ijlm}=g\epsilon_{ijp_1\cdots p_{n-3}u}\epsilon_{lmq_1\cdots q_{n-3}v}\varphi^{uv}g^{p_1q_1}\cdots g^{p_{n-3}q_{n-3}},$$

where φ^{uv} is symmetric, as a necessary and sufficient condition for a space to be conformally flat. He uses this theorem to show that spaces which satisfy (9) are those that give an Einstein static spherically symmetric universe. Hence the only types of universes allowed by the wave geometry theory of cosmology are the Einstein or the DeSitter.

Takeno [the same J. 8, 223–237 (1938)] has obtained solutions for equations (6) and (8) and computed u^i on the assumption that in the certain coordinate systems the radial motion of nebulae is unobserved. The u^i thus obtained depend on two parameters and for a given choice of these parameters state the nature of the motion at any point in the universe. Itimaru's present paper is the continuation of a previous one [the same J. 8, 239–248 (1938)]. There it was shown in the usual manner how the u^i derived from (6) gives the red shift for a suitable choice of these parameters and that the u^i derived from (8) corresponds to a static universe. Now Itimaru interprets the vector u^i as the momentum energy vector of a fluid and computes the mass of the universe in terms of the integral of the time component of the vector u^i. The reason for equating the velocity vectors of the fluid with its momentum energy vector is not given. The fact that the divergence of u^i does not vanish is "explained" by the fact that radiation has not been taken into account and this will allow a creation and annihilation of matter. Finally Iwatsuki gives reasons for discarding the proper value equation $ds\psi=\mu\psi$. *A. H. Taub.*

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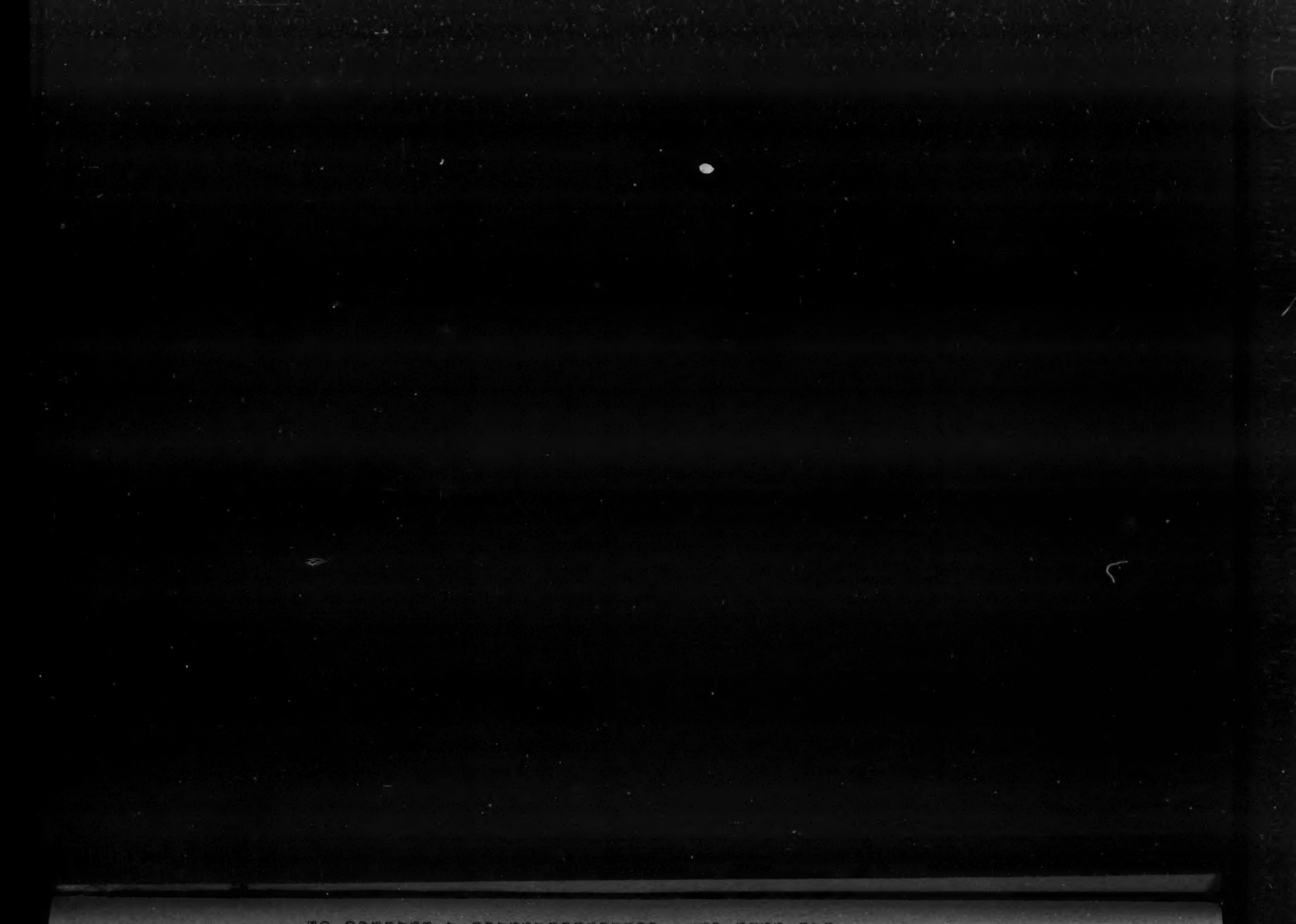
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